

ANNUAL REPORT

A DIRECT APPROACH TO THE DESIGN
OF LINEAR MULTIVARIABLE SYSTEMS

NGL-

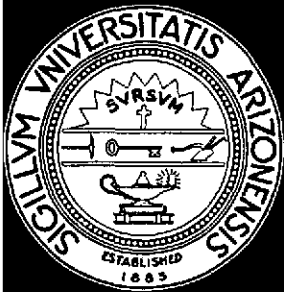
NASA Grant 03-002-006

June, 1974

(NASA-CR-138516) A DIRECT APPROACH TO
THE DESIGN OF LINEAR MULTIVARIABLE
SYSTEMS Annual Report (Arizona Univ.,
Tucson.) 150 p HC \$10.50 CSCL 20F

N74-27192

Unclas
G3/23 41188



ENGINEERING EXPERIMENT STATION
COLLEGE OF ENGINEERING
THE UNIVERSITY OF ARIZONA
TUCSON, ARIZONA

A DIRECT APPROACH TO THE DESIGN OF
LINEAR MULTIVARIABLE SYSTEMS

by

Bajarang Lall Agrawal

NASA 03-002-006

ANNUAL REPORT

June 1974

Donald G. Schultz
Principal Investigator

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	vi
ABSTRACT	vii
 CHAPTER	
1 INTRODUCTION	1
2 STRUCTURE OF MULTIVARIABLE SYSTEMS	6
2.1 Introduction and Organization of the Chapter	6
2.2 System Representation, Definitions and Notations	7
2.3 Luenberger Canonical Transformation	10
2.4 Structure Theorem of Wolowich and Falb	15
2.5 The Necessary and Sufficient Conditions for Exact Model Matching	26
2.6 The Problem of Decoupling	34
2.7 Summary and Conclusions	42
3 THE DESIGN EQUATION	44
3.1 Introduction and Organization of the Chapter	44
3.2 The Generalized Error Coefficients for Multivariable Systems	45
3.3 The Derivatives of the Feedback Invariant Matrix $R(s)$	49
3.4 Relation Among Component Matrices	52
3.5 Summary	58
4 A DESIGN METHOD FOR MULTIVARIABLE SYSTEMS	60
4.1 Introduction and Organization of the Chapter	60
4.2 Notation and Identity	60
4.3 Design Method I	65
4.3.1 Proof of Theorem 4.1	70
4.4 Step-by-Step Design Procedure	75
4.5 Example 4.1	77
4.5.1 The Problem of Decoupling	83
4.5.2 The General Case of Model Matching	86

(PAGES ii THRU iii)
BLANK

iv

TABLE OF CONTENTS (Continued)

	Page
4.6 Conclusion	100
5 THE POLE FIXING METHOD	103
5.1 Introduction and Organization of the Chapter	103
5.2 Need for Pole Fixing	103
5.3 The Pole Fixing Method	105
5.4 Summary of the Design Procedure	119
5.5 Example 5.1	121
5.6 Special Cases	132
5.6.1 The Case of Triangular Decoupling	132
5.6.2 The Minimum Constraint Case	135
5.7 Summary	136
6 SUMMARY AND CONCLUSION	137
6.1 Summary	137
6.2 Further Research	139
SELECTED BIBLIOGRAPHY	142

LIST OF ILLUSTRATIONS

Figure		Page
4.1	Block Diagram of the System	79
4.2	Comparison of Time Responses in Example 4.1	96
4.3	Compensated System	101
5.1	Comparison of Time Responses in Example 5.1	133

ABSTRACT

Design of multivariable systems is considered and design procedures are formulated in the light of the most recent work on model matching. The word model matching is used exclusively here to mean matching the input-output behavior of two systems. In the frequency domain the term is used to indicate the comparison of two transfer matrices containing transfer functions as elements. The design of multivariable systems is particularly complicated by the fact that the transfer matrix is a ratio of two polynomial matrices. The use of state variable feedback not only affects the pole positions, but also alters the positions of most of the zeros of individual transfer functions due to interaction. Because of these complexities, non-interaction has been one of the main criteria of design in the past.

This study concentrates on design methods where non-interaction is not used as a criteria. Non-interacting design is obtained as a special case of the more general interacting design. Two design methods are considered. In the first method the design is based solely upon the specification of generalized error coefficients for each individual transfer function of the overall system transfer matrix. The main disadvantage of such a design is that it does not take into consideration the transient response of the system, as the concept of error coefficients is based solely upon steady state behavior. Thus, the

transient response of the system may not be satisfactory. However, if the transient response is satisfactory, the main advantages of the first method is that it is simple and gives a realizable solution to the problem.

The second design method is called the pole fixing method because in it all the system poles are fixed at preassigned positions. In addition, the zeros of terms either above or below the diagonal can be partially fixed via steady state error coefficients. This is a considerable improvement over the conventional decoupled design where it might not even be possible to decouple the system. Even if the system were decouplable, some of the system poles could be uncontrollable. However, designing the system using the pole fixing method requires that certain sufficiency conditions be met. In many cases these sufficiency conditions may be satisfied by a name change in the output vector.

An example is worked to demonstrate the use of both design methods. The special case of triangular decoupling and minimum constraints are discussed.

CHAPTER 1

INTRODUCTION

The problem of designing a linear multivariable system has long occupied the minds of control engineers, especially recently in view of its vast application in engineering and related fields. Most of the earlier work is centered on designing noninteracting systems. Efforts started as early as 1950 and continued through most of the 1960's [Boksenbom and Hood, 1949; Kavanagh, 1956, 1957, 1958; Horowitz, 1960; Morgan, 1964; Rekasius, 1965]. Falb and Wolowich [1967] first formulated and proved necessary and sufficient conditions for determining whether or not linear state variable feedback (l.s.v.f.) can decouple a multivariable system. They also gave a design procedure for placing the poles of the decoupled system at desired places. Then Gilbert [1969] gave a physical interpretation to the results of Falb and Wolowich and explained the notion of feedback invariance. Gilbert's work provided the complete answer to the problem of decoupling a multivariable system using l.s.v.f.

But the problem of design of multivariable systems was far from solved, except for the special case of decoupling. Luenerger [1967] devised a transformation matrix which puts the state equations of multivariable system into a canonical form, similar to phase variable form for the scalar case of single input-single output systems. Wolowich and Falb [1969] made use of the canonical transformation

devised by Luenberger [1967] and came up with startling results on the structure of multivariable systems.

While a lot of attention was devoted to using dynamic compensation for multivariable systems [Wolowich, 1973; Moore and Silverman, 1971; Morse and Wonham, 1970; Pearson and Ding, 1969; Silvisky, 1969] studies continued utilizing l.s.v.f. for design of interacting control systems [Ferg, 1971; Heinz, 1968; Anderson and Luenberger, 1967]. The design of interacting multivariable systems is given the name Model Matching. In view of model matching some pertinent questions arise.

Does there exist a set of feedback and gain matrices $\{F, G\}$ such that when applied to a given system the input-output behavior of the system characterized by the frequency domain transfer matrix $T(s)$ matches a pre-specified input-output behavior. Wolowich [1972] answered this question in great detail. He utilized the structure theorem [Wolowich and Falb, 1969] and for the first time developed a complete set of feedback invariants for multivariable systems. He also gave necessary and sufficient conditions for Exact Model Matching, and outlined the procedure to find the pair $\{F, G\}$, if the necessary and sufficient conditions were met.

However, the main design question of what to do if the necessary and sufficient conditions are not met remains unanswered. Unfortunately, this is the most important problem a designer is likely to face. The necessary and sufficient conditions for model matching do not give any hint as to what can be expected from the system, or what changes need to be made in the model transfer matrix to ensure

that the necessary and sufficient conditions are satisfied. The designer is thus left with no guidelines with which to proceed.

In this study a completely new and basic approach is taken to the design of multivariable systems. The approach provided at least a partial answer to the above questions. The approach taken in this study is based upon generalized error coefficients. The design specifications for multivariable systems could either be expressed directly in terms of generalized error coefficients or could be specified as transfer functions. In the latter case the transfer function is represented in terms of its Maclaurin series so that the coefficients of the successive powers of s have a direct relationship with the generalized error coefficients.

A design based solely on error coefficients may not be satisfactory for the two reasons. First, only a specified number of error coefficients can be incorporated into the design. Secondly, the error coefficients describe the system input-output behavior only after sufficient time has elapsed for system transients to decay to insignificant amplitudes. Since the pole positions are not known until after the design is complete, decaying transients are not ensured in advance.

A second design technique called pole fixing design method takes care of the above uncertainty by pre-specifying all the pole positions in advance. The main feature of this method is that in addition to arbitrarily fixing the poles, it maintains enough freedom to specify error coefficients of one side of diagonal terms in the transfer matrix,

thus in turn, indirectly specifying system zeros. This method has the following distinct advantages over the conventional decoupled design.

1. All the off-diagonal terms are not forced to zero,
2. There are no uncontrollable poles, and
3. Those systems which cannot be decoupled could very well be designed using pole fixing method.

A disadvantage of the pole fixing method is the requirement that the system satisfy certain additional sufficiency conditions. However, these sufficiency conditions are variant under the name change of inputs and outputs and are also dependent upon the output matrix of the system. Thus, in many cases, the sufficiency conditions could be satisfied by simple changes in the system output matrix.

This study has been organized as follows. In Chapter 2 notations are introduced and a brief treatment of background material is provided. This includes a discussion of the feedback invariants and structure of the multivariable systems. The representation of a transfer matrix as a ratio of two polynomial matrices is used to derive the central result of this study in Chapter 3. The relation between the error coefficients and the components of polynomial matrices representing the transfer matrix is obtained. This relation produces a series of equations which can be evaluated in sequence.

Additional notations are given in Chapter 4 as a means of denoting the submatrices of a component matrix. An identity is introduced to partition the multiplication of two matrices into multiplication of its submatrices. The design freedom is illustrated with the

help of Theorem 4.1. The proof of the theorem results in a step-by-step design procedure for the design of multivariable systems utilizing the generalized error-coefficients as the design criteria. An example is used to illustrate the design steps.

Chapter 5 develops the pole fixing design method. The shortcomings of the error coefficient design method are overcome by the pole fixing method. Once again the design constraints are introduced via Theorem 5.1. A step-by-step design procedure is given for a quick and easy reference and the example of Chapter 4 is reworked using pole fixing method.

This study is concluded with Chapter 6 where the results of this study are summarized and further research is suggested.

CHAPTER 2

STRUCTURE OF MULTIVARIABLE SYSTEMS

2.1 Introduction and Organization of the Chapter

This chapter provides a review of existing work on the structure and exact model matching of multivariable system. Section 2.2 describes the system representation in state variable form followed by some definitions and notations. Section 2.3 describes a transformation [Luenberger, 1967] which produces a phase variable representation for multivariable systems. The above transformation is used in Section 2.4 where the structure theorem of Wolowich and Falb [1969] is explained and proved. The advantage of such a structure is that it separates the system input output behavior transfer matrix in two parts, one which is invariant under state variable feedback and, second, which is almost completely dependent upon state variable feedback. The above property is used to derive necessary and sufficient conditions for exact model matching [Wolowich, 1972], as briefly discussed in Section 2.5.

The past work on the decoupling of multivariable systems has intentionally not been discussed because almost all the information about decoupling is provided by the structure theorem. Hence, in Section 2.6 the problem of decoupling is discussed as a special case of the problem of exact model matching. It is shown how all the relevant information about decoupling can be obtained from the use of

structure theorem and from the necessary and sufficient conditions for exact model matching of the multivariable systems.

Finally, this chapter is concluded with a discussion of the advantages as well as hazards of attempting to use the above mentioned necessary and sufficient conditions for model matching.

2.2 System Representation, Definitions and Notations

Throughout the study it is assumed that the linear multivariable time invariant open loop system is represented in state variable form by following well-known equations.

$$\begin{aligned}\dot{X} &= AX + BU \\ Y &= CX\end{aligned}\tag{2.1}$$

where

$$\begin{array}{ll} X \in E^n & \text{system state vector} \\ Y \in E^m & \text{system output vector} \\ U \in E^m & \text{system input vector} \end{array}$$

The matrices A, B, and C are constant system matrices of appropriate dimensions. The following state variable feedback is considered

$$U = FX + GR\tag{2.2}$$

where F is $m \times n$ and G is $m \times m$ constant matrices and R is new reference control input vector. With the state variable feedback as specified in

Eq. (2.2), closed loop state variable representation of the system is given by

$$\begin{aligned}\dot{X} &= (A + BF)X + BGR \\ Y &= CX\end{aligned}\tag{2.3}$$

While the system is described completely in the time domain, much of the work contained here has been transformed into frequency domain. The following definitions and notations prove useful.

Definition 2.1 Transfer Matrix: A transfer matrix is any matrix whose elements are transfer functions representing the input-output behavior of a single input - single output system in the frequency domain.

The transfer matrix associated with the closed loop system of Eq. (2.3) is denoted by $T(s)$ and is given by

$$T(s) = C(sI - A - BF)^{-1}BG\tag{2.4}$$

Definition 2.2 Proper and Strictly Proper Transfer Matrices:

A transfer matrix is called proper (strictly proper) if for every transfer function element, the degree of the numerator polynomial is less than or equal to (strictly less than) the degree of the corresponding denominator polynomial.

The transfer matrix of the closed loop system of Eq. (2.3) given by Eq. (2.4) is a strictly proper transfer matrix because there is no direct feed forward from inputs to outputs.

Definition 2.3 Polynomial Matrix: A polynomial matrix is any matrix whose elements are polynomials of degree greater than or equal to zero.

A polynomial matrix is usually denoted by capital letter with explicit s dependence unless otherwise mentioned. The transfer matrix of Eq. (2.4) can be described as a product of two polynomial matrices as $T(s) = R(s)P(s)^{-1}$. Here $P(s)^{-1}$ represents the inverse of the polynomial matrix $P(s)$.

Unless otherwise mentioned, the i^{th} row and i^{th} column of a constant matrix is denoted by a subscript i and superscript i , respectively; viz., b_i denotes i^{th} row of B matrix and b^j denotes j^{th} column of B matrix. The i^{th} row of a non constant matrix is denoted by a subscript on capital letter, viz., $P_i(s)$ denotes i^{th} row of $P(s)$.

The following notations are equivalent for any constant or polynomial matrices.

$$B = [b_{ij}]$$

$$P(s) = [p_{ij}(s)] \quad (2.5)$$

where b_{ij} denotes the element at the cross section of i^{th} row and j^{th} column of B matrix and $p_{ij}(s)$ denotes polynomial at the cross section of i^{th} row and j^{th} column of polynomial matrix $P(s)$. The determinants of a constant matrix B and a non constant matrix $P(s)$ are denoted as follows

$$\text{determinant } B = |B|$$

$$\text{determinant } P(s) = |P(s)| \quad (2.6)$$

2.3 Luenberger Canonical Transformation

Various canonical transformations are available for multivariable systems. The one given by Luenberger [1967] is particularly useful because the matrices representing the transformed system can be partitioned into submatrices which have a phase variable form. The property of the phase variable representation of a system is that for the scalar case of single input-single output system, all the relevant information about the input output behavior of the system is contained in the last row of system matrices. The system transfer-function can be written by inspection of system matrices.

Even though the above does not hold in the multivariable case, i.e., the transfer matrix cannot be written by inspection, the transformation is useful because it separates the system transfer matrix in an invariant part and another part which arbitrarily depends upon state variable feedback except for its form. This is similar to the scalar case where it is well known that zeros of transfer function are invariant and the poles can be arbitrarily placed by state variable feedback. Only the order of system cannot be increased.

To find the transformation matrix and the transformed system, consider the system described in Eq. (2.1). It is assumed that the input matrix B has rank m (otherwise, the m inputs are not independent) and the output matrix C has rank m (otherwise, the m outputs are

not independent). It is also assumed that the system described in Eq. (2.1) denoted by $\{A, B, C\}$ is controllable, i.e., the $n \times m$ controllability matrix

$$[B: AB: \dots A^{n-1}B] \quad (2.7)$$

has rank n .

That is to say the above matrix has at least n independent vectors; however, in general there may be more than one set of n independent vectors.

The following algorithm finds a transformation matrix denoted by Q which has the properties as described above.

Step 1: Find the first n independent columns of the controllability matrix given in Eq. (2.7). Let the B matrix be represented by

$$B = [b^1 \ b^2 \ \dots \ b^m]$$

where b^i represents i^{th} column of B matrix.

Step 2: Rearrange the n independent columns found in Step 1 in the following manner

$$[b^1, Ab^1, \dots, A^{\sigma_1-1}b^1, b^2, \dots, A^{\sigma_2-1}b^2, \dots, A^{\sigma_m-1}b^m] \quad (2.8)$$

where as indicated $A^j b^i$ is the i^{th} column of $n \times m$ matrix $A^j B$. Notice that the above rearrangement groups together σ_i independent vectors which are related to i^{th} column b^i of the input matrix B . Thus, informally speaking, the n^{th} order multivariable system has been divided into m subsystems with the i^{th} subsystem having order σ_i .

Notice that

$$\sigma_i \geq 1 \quad \text{for } i = 1 \dots m,$$

and

$$\sum_{i=1}^m \sigma_i = n.$$

Step 3: Define the integers d_K and the $1 \times n$ row vectors ℓ_K as follows:

$$d_K = \sum_{i=1}^K \sigma_i \quad \text{for } K = 1, 2, \dots, \quad (2.9)$$

$\ell_K = d_K$ th row of the inverse of the matrix

in Eq. (2.8)

Step 4: Form the $n \times n$ transformation matrix Q as follows:

$$Q = \begin{bmatrix} \ell_1 \\ \ell_1 A \\ \vdots \\ \ell_1 A^{\sigma_1-1} \\ \ell_2 \\ \ell_2 A \\ \vdots \\ \ell_2 A^{\sigma_2-1} \\ \vdots \\ \ell_m A^{\sigma_m-1} \end{bmatrix} \quad (2.10)$$

If the transformation $\hat{X} = QX$ is used, where Q is the nonsingular matrix as given above and \hat{X} is the new state vector, then the new system equations are given by

$$\begin{aligned}\dot{\hat{X}} &= \hat{A}\hat{X} + \hat{B}U \\ Y &= \hat{C}\hat{X}\end{aligned}\tag{2.11}$$

where

$$\begin{aligned}\hat{A} &= QAQ^{-1} & A &= Q^{-1}\hat{A}Q \\ \hat{B} &= QB & B &= Q^{-1}\hat{B} \\ \hat{C} &= CQ^{-1} & C &= \hat{C}Q\end{aligned}\tag{2.12}$$

and the state variable feedback is given by

$$U = \hat{F}\hat{X} + \hat{G}R\tag{2.13}$$

where

$$\hat{F}Q = F \quad \text{and} \quad \hat{G} = G\tag{2.14}$$

As mentioned above, the main advantage of such a transformation is that the matrices A and B take a special form as shown below.

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}\tag{2.15}$$

where A_{ij} for $i=j$ is a $\sigma_i \times \sigma_i$ matrix in phase variable form as shown below.

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{bmatrix} \quad (2.16)$$

where $*$'s denote non zero terms in general.

For $i \neq j$, A_{ij} is $\sigma_i \times \sigma_j$ matrix with all but its σ_i th row identically equal to zero. The matrix \hat{B} in Eq. (2.8) takes the following special form.

$$\hat{B} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \hat{b}_{d_1,1} & \hat{b}_{d_1,2} & \dots & \hat{b}_{d_1,m} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \hat{b}_{d_2,1} & \hat{b}_{d_2,2} & \dots & \hat{b}_{d_2,m} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ \hat{b}_{d_m,1} & \hat{b}_{d_m,2} & \dots & \hat{b}_{d_m,m} \end{bmatrix} \quad (2.17)$$

where d_K 's are as defined in Eq. (2.9). Thus, only d_K th rows of \hat{B} are non zero.

It is worth noticing at this point that the transformation matrix Q obtained in Eq. (2.10) is not the only transformation matrix which places \hat{A} and \hat{B} in the form given in Equations (2.15) and (2.17) respectively. In general there may be other arrangements for picking n independent columns out of the controllability matrix of Eq. (2.7) which lead to a transformation matrix so that \hat{A} and \hat{B} have the above mentioned forms. That is to say, in general the σ 's (roughly speaking σ_i is the order of i^{th} subsystem) are not unique, but in the absence of any other general method for finding the transformation matrix Q , the algorithm described in this section is used throughout this study.

2.4 Structure Theorem of Wolowich and Falb

To be able to better understand the structure theorem it is necessary to give a few definitions introduced by Wolowich and Falb [1969] in connection with the structure theorem.

Definition 2.4 S^σ : S^σ is defined as the $m \times m$ diagonal matrix as given below.

$$S^\sigma = \begin{bmatrix} s^{\sigma 1} & 0 & 0 & \dots & 0 \\ 0 & s^{\sigma 2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & s^{\sigma m} \end{bmatrix} \quad (2.18)$$

Thus the i^{th} diagonal entry of the matrix S^σ is $s^{\sigma i}$.

Definition 2.5 S(s): S(s) is defined as the following nxm matrix of single term monic polynomials.

$$S(s) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ s^{\sigma_1-1} & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & s^{\sigma_2-1} & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ & & & \vdots \\ 0 & 0 & & s^{\sigma_m-1} \end{bmatrix} = \begin{bmatrix} S_1(s) & 0 & & 0 \dots 0 \\ 0 & S_2(s) & & 0 \dots 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \dots S_m(s) \end{bmatrix} \quad (2.19)$$

where

$$S_i(s) = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{\sigma_i-1} \end{bmatrix}$$

is a $\sigma_i \times 1$ column vector.

Definition 2.6 \tilde{A} : \tilde{A} is defined as mxn matrix consisting of the m ordered d_K th rows of $\hat{A} = [\hat{a}_{ij}]$. Thus, \tilde{A} is given as follows:

$$\tilde{A} = \begin{bmatrix} \hat{a}_{d_1,1} & \hat{a}_{d_1,2} & \dots & \hat{a}_{d_1,n} \\ \hat{a}_{d_2,1} & \hat{a}_{d_2,2} & \dots & \hat{a}_{d_2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{d_m,1} & \hat{a}_{d_m,2} & \dots & \hat{a}_{d_m,n} \end{bmatrix} \quad (2.20)$$

Definition 2.7 \tilde{B} : \tilde{B} is defined as the $m \times m$ matrix consisting of the m ordered d_K the rows of $\hat{B} = [\hat{b}_{ij}]$. Thus, \tilde{B} is given as follows:

$$\tilde{B} = \begin{bmatrix} \hat{b}_{d_1,1} & \hat{b}_{d_1,2} & \dots & \hat{b}_{d_1,m} \\ \hat{b}_{d_2,1} & \hat{b}_{d_2,2} & \dots & \hat{b}_{d_2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}_{d_m,m} & \hat{b}_{d_m,2} & \dots & \hat{b}_{d_m,m} \end{bmatrix} \quad (2.21)$$

Equivalently if a matrix E is defined as follows:

$$E = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & \\ \vdots & \vdots & & & & \\ \cdot & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & & & & \cdot \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & 1 \end{bmatrix} \quad (2.22)$$

Then clearly

$$\hat{B} = E\tilde{B} \quad (2.23)$$

In view of above definitions and the form of transformed matrices \hat{A} and \hat{B} as given in Equations (2.15) and (2.17), respectively,

notice that $\hat{B}\hat{F}$ has all but the m ordered d_K th rows zero. Thus, $\hat{A} + \hat{B}\hat{F}$ has exactly the same form as \hat{A} . Let

$$\Phi = [\phi_{ij}] = \hat{A} + \hat{B}\hat{F} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ \vdots & & & \\ \phi_{m1} & \phi_{m2} & \cdots & \phi_{mm} \end{bmatrix} \quad (2.24)$$

Then for $i=j$, ϕ_{ij} is $\sigma_i \times \sigma_j$ matrix in phase variable form as shown in Eq. (2.16) and for $i \neq j$, ϕ_{ij} is a $\sigma_i \times \sigma_j$ matrix with all but last row identically zeros. Thus, Φ has exactly the same form as \hat{A} as given in Eq. (2.15).

Hence, as in Eq. (2.20), define $\tilde{\Phi} = \tilde{A} + \tilde{B} \hat{F}$ as a $m \times n$ matrix consisting of the m ordered d_K th row of the matrix Φ as defined above in Eq. (2.24) and partition it into m^2 row vectors as follows:

$$\tilde{\Phi} = \tilde{A} + \tilde{B} \hat{F} = \begin{bmatrix} \tilde{\phi}_{11} & \tilde{\phi}_{12} & \cdots & \tilde{\phi}_{1m} \\ \tilde{\phi}_{21} & \tilde{\phi}_{22} & \cdots & \tilde{\phi}_{2m} \\ \vdots & & & \\ \tilde{\phi}_{m1} & \tilde{\phi}_{m2} & \cdots & \tilde{\phi}_{mm} \end{bmatrix} \quad (2.25)$$

Then, $\tilde{\phi}_{ij}$ is $1 \times \sigma_j$ row vector obtained by partitioning the d_i th row of Φ into m row vectors. Thus

$$[\tilde{\phi}_{i1} \quad \tilde{\phi}_{i2} \quad \cdots \quad \tilde{\phi}_{im}] = [\phi_{d_i,1} \quad \phi_{d_i,2} \quad \cdots \quad \phi_{d_i,n}] \quad (2.26)$$

The structure theorem is now derived which essentially separates the system input-output transfer matrix into invariant and dependent parts with respect to state variable feedback. As mentioned earlier in the Definition (2.1), for the multivariable system of Eq. (2.1) with feedback given in Eq. (2.2), the input-output transfer matrix is given by Eq. (2.4). Substitution for A, B, and C from the set of Equations (2.12) in Eq. (2.4) gives

$$\begin{aligned} T(s) &= \hat{C}Q(sI - Q^{-1}\hat{A}Q - Q^{-1}\hat{B}\hat{F})^{-1} Q^{-1}\hat{B}G \\ &= \hat{C}(sI - \hat{A} - \hat{B}\hat{F}Q^{-1})\hat{B}G \\ &= \hat{C}(sI - \hat{A} - \hat{B}\hat{F})^{-1} \hat{B}G \end{aligned}$$

Substitution for $\hat{A} + \hat{B}\hat{F}$ from Eq. (2.24) in above gives

$$T(s) = \hat{C}(sI - \hat{\Phi})^{-1} \hat{B}G \quad (2.27)$$

where

$$\hat{F} = F Q^{-1} \quad (2.28)$$

It is claimed by Falb and Wolowich [1969] that the system transfer matrix of Eq. (2.28) can be written as follows.

$$T(s) = \hat{C} S(s) [(\hat{B}G)^{-1} \{s^\sigma - \hat{\Phi} S(s)\}]^{-1} \quad (2.29)$$

or, equivalently

$$\hat{C}(sI - \hat{\Phi})^{-1} \hat{B}G = \hat{C} S(s) [S^\sigma - \tilde{\Phi}S(s)]^{-1} \tilde{B}G \quad (2.30)$$

where S^σ , $S(s)$, $\tilde{\Phi}$ are as defined in Eqs. (2.18), (2.19), and (2.25), respectively. In their proof of the claim, \hat{B} is not as general as given by Eq. (2.17). The proof given below is more clear and is consistent with the notion introduced in this section.

Post-multiply both sides of Eq. (2.30) by G^{-1} . Then, to prove Eq. (2.30), it suffices to prove the following:

$$(sI - \Phi)^{-1} \hat{B} = S(s) [S^\sigma - \tilde{\Phi}S(s)]^{-1} \tilde{B}$$

After a trivial manipulation, it can be shown that it suffices to prove the following:

$$(sI - \Phi) S(s) = \hat{B} \tilde{B}^{-1} [S^\sigma - \tilde{\Phi}S(s)]$$

After substituting for \hat{B} from Eq. (2.23), it now becomes necessary to prove only the following:

$$(sI - \Phi) S(s) = E[S^\sigma - \tilde{\Phi}S(s)] \quad (2.31)$$

Expansion of the left hand side of the above equation results in

$$\begin{aligned}
 (sI - \Phi)S(s) &= \begin{bmatrix} sI - \phi_{11} & -\phi_{12} & \dots & -\phi_{1m} \\ -\phi_{21} & sI - \phi_{22} & \dots & -\phi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_{m1} & -\phi_{m2} & \dots & sI - \phi_{mm} \end{bmatrix} \begin{bmatrix} S_1(s) & 0 & \dots & 0 \\ 0 & S_2(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_m(s) \end{bmatrix} \\
 &= \begin{bmatrix} (sI - \phi_{11})S_1(s) & -\phi_{12}S_2(s) & \dots & -\phi_{1m}S_m(s) \\ -\phi_{21}S_1(s) & (sI - \phi_{22})S_2(s) & \dots & -\phi_{2m}S_m(s) \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_{m1}S_1(s) & -\phi_{m2}S_2(s) & \dots & (sI - \phi_{mm})S_m(s) \end{bmatrix}
 \end{aligned}
 \tag{2.32}$$

Now examine the (i,j) th block matrix in the above equation for $i=j$

$$\begin{aligned}
 (sI - \phi_{ij})S_i(s) &= \begin{bmatrix} s & -1 & 0 & \dots & 0 & 0 \\ 0 & s & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & s & -1 \\ * & * & * & \dots & * & s - \phi_{di,di} \end{bmatrix} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\sigma_i-2} \\ s^{\sigma_i-1} \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ s^{\sigma_i} - \tilde{\phi}_{ij}S_i(s) \end{bmatrix}
 \end{aligned}$$

The above is a $\sigma_i \times 1$ vector with all but the last element non zero. For $i \neq j$

$$\begin{aligned}
 -\tilde{\Phi}_{ij} S_j(s) &= \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & * & * \end{bmatrix} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\sigma_j-2} \\ s^{\sigma_j-1} \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\tilde{\Phi}_{ij} S_j(s) \end{bmatrix}
 \end{aligned}$$

Again, this is a $\sigma_i \times 1$ column vector.

After substituting for these block matrices in Eq. (2.32), the following is readily obtained.

$$(sI - \Phi)S(s) \approx \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ s^{\sigma_1} - \tilde{\phi}_{11}S_1(s) & -\tilde{\phi}_{12}S_2(s) & & -\tilde{\phi}_{1m}S_m(s) \\ 0 & 0 & & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ 0 & 0 & & \\ -\tilde{\phi}_{21}S_1(s) & s^{\sigma_2} - \tilde{\phi}_{22}S_2(s) & \dots & -\tilde{\phi}_{2m}S_m(s) \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ -\tilde{\phi}_{m1}S_1(s) & -\tilde{\phi}_{m2}S_2(s) & \dots & s^{\sigma_m} - \tilde{\phi}_{mm}S_m(s) \end{bmatrix}$$

$$= E \begin{bmatrix} s^{\sigma_1} - \tilde{\phi}_{11}S_1(s) & -\tilde{\phi}_{12}S_2(s) & \dots & -\tilde{\phi}_{1m}S_m(s) \\ -\tilde{\phi}_{21}S_1(s) & s^{\sigma_2} - \tilde{\phi}_{22}S_2(s) & \dots & -\tilde{\phi}_{2m}S_m(s) \\ \vdots & \vdots & & \\ -\tilde{\phi}_{m1}S_1(s) & -\tilde{\phi}_{m2}S_2(s) & \dots & s^{\sigma_m} - \tilde{\phi}_{mm}S_m(s) \end{bmatrix}$$

(2.33)

where E is as defined in Eq. (2.22).

The above Eq. (2.33) can be written as follows:

$$(sI - \Phi)S(s) = E \left\{ \begin{bmatrix} s^{\sigma_1} & 0 & 0 & \dots & 0 \\ 0 & s^{\sigma_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & s^{\sigma_m} \end{bmatrix} - \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \dots & \tilde{\Phi}_{1m} \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} & \dots & \tilde{\Phi}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Phi}_{m1} & \tilde{\Phi}_{m2} & \dots & \tilde{\Phi}_{mm} \end{bmatrix} \begin{bmatrix} S_1(s) & 0 & \dots & 0 \\ 0 & S_2(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_m(s) \end{bmatrix} \right\}$$

and substituting for S^σ and $S(s)$ from Eqs. (2.18) and (2.19), respectively, one gets

$$(sI - \Phi) = E\{S^\sigma - \tilde{\Phi}S(s)\}$$

The above is exactly the result claimed in Eq. (2.31). Thus, the claim of Falb and Wolowich [1969] is demonstrated to be true for \hat{B} of Eq. (2.17), which is more general than the one used by them.

It is thus established that system input output transfer matrix $T(s)$ can be written as follows:

$$T(s) = \hat{C} S(s) [(\tilde{B}G)^{-1}\{S^\sigma - \tilde{\Phi}S(s)\}]^{-1}$$

After substituting from Eq. (2.25) for $\tilde{\Phi}$ one obtains, from above

$$T(s) = \hat{C} S(s) [(\tilde{B}G)^{-1} \{S^\sigma - (\tilde{A} + \tilde{B}\hat{F}) S(s)\}]^{-1}$$

that is,

$$T(s) = R(s) P(s)^{-1} \quad (2.34)$$

where

$$R(s) = \hat{C} S(s) \quad (2.35)$$

is a $m \times m$ polynomial matrix independent of state variable feedback and thus forms the invariant part of transfer matrix for any state variable feedback of the type shown in Eq. (2.2), and

$$P(s) = [(\tilde{B}G)^{-1} \{S^\sigma - (\tilde{A} + \tilde{B}\hat{F}) S(s)\}] \quad (2.36)$$

is also a $m \times m$ polynomial matrix. Since $P(s)$ is a function of both F and G , $P(s)$ takes into account all the effects of the state variable feedback of Eq. (2.2).

It is worth noticing at this point that $P(s)$ not only accounts for all the feedback and gain, but has a unique form for a given set of sigmas. Recall the sigmas are fixed (even though not unique). Also, notice that the coefficient of the highest order term in each (i,j) th polynomial is given by the corresponding (i,j) th term in the matrix

$(\tilde{B}G)^{-1}$. Thus, by choosing G properly, coefficients of highest order term in each polynomial of $P(s)$ can be arbitrarily manipulated.

The coefficients of all other lower degree terms in the polynomials of $P(s)$ can be arbitrarily placed by proper choice of feedback matrix F because \tilde{B} is a nonsingular matrix.

Thus, the conclusion is that, except for its form, the polynomial matrix $P(s)$ is almost completely arbitrary. The only exception is that the matrix coefficients consisting of the highest degree polynomial in each column of $P(s)$ must be a nonsingular matrix because that matrix is $(\tilde{B}G)^{-1}$ which is a nonsingular matrix. The above property of a polynomial matrix is referred to as being column proper by Wolowich [1972].

2.5 The Necessary and Sufficient Conditions for Exact Model Matching

The term model matching is more easily explained in the scalar case of single input-single output systems. Given a scalar system (plant) and a scalar model if by some means (here by state variable feedback), the plant input-output transfer function can be made equal to the model transfer function, then the system (plant with feedback) is said to be matched with the model because their input output behavior is identical. In the scalar case it is well known that

- 1) The zeros of plant transfer function are feedback invariants.
- 2) The order of the closed loop plant can not be increased by state variable feedback.

- 3) If the plant is controllable, then the poles of the plant transfer function can be arbitrarily placed by using state variable feedback.
- 4) The closed loop gain can be adjusted at desired level.

Knowing the above feedback invariants and dependents, the necessary and sufficient conditions for exact model matching are easily obtained for the above scalar case and are well known. Namely, the system can be matched to the desired closed loop transfer function, that is, the model, by state variable feedback alone if, and only if,

- 1) The model has zeros at the same place as that of the plant.
- 2) The pole zero excess of the model is the same as that of the plant.
- 3) If the model has zeros other than plant zeros, they must cancel with the model poles.

In the scalar case the knowledge of the above is necessary and sufficient conditions for model matching lead directly to a design procedure. Even though the term exact model matching for multivariable system is thought of as just an extension of the scalar case, it is actually more than this, i.e., instead of matching two transfer functions, one has to match two transfer matrices. Several additional definitions are needed to extend the discussion to the multivariable case.

Definition 2.8 Unimodular Matrix: A polynomial matrix is called unimodular if the determinant of a matrix is a non zero polynomial of degree zero, i.e., is a scalar.

Definition 2.9 Division of a Polynomial Matrix: A polynomial matrix $R(s)$ is said to be right [left] divisible by another matrix $H(s)$ if $R(s) H(s)^{-1}$ [$H(s)^{-1} R(s)$] is also a polynomial matrix. (Consequently, $H(s)$ is called right (left) divisor of $R(s)$.)

Definition 2.10 Relatively Prime Polynomials: Two or more polynomials (taken together) are called relatively prime, if there does not exist any polynomial of degree greater than zero which divides all the polynomials simultaneously.

Definition 2.11 Relatively Prime Polynomial Matrices: Two polynomial matrices are said to be relatively right [left] prime, if there does not exist any non-unimodular matrix which right [left] divides both the matrices. (Notice that any unimodular matrix divides any polynomial matrix and, hence, while making a test for relative primeness only non-unimodular matrices may be considered.)

The necessary and sufficient conditions for exact model matching are given in the following two theorems which are due to Wolowich [1971, 1972].

Theorem 2.1. A $(m \times m)$ rational strictly proper transfer matrix can always be factored as the product.

$$T(s) = R(s) P^{-1}(s) \quad (2.37)$$

where $R(s)$ and $P(s)$ are relatively right prime polynomial matrices, respectively, in the Laplace operators, with $P(s)$ non singular and column proper.

A formal constructive proof of this theorem is given in [Wolowich, 1971] and is not repeated here.

Theorem 2.2. Consider the system given in Eq. (2.1) with $\{A, B\}$ controllable and B of full rank $m \leq n$, and let the the model transfer function be given by

$$T_m(s) = R_m(s) P_m^{-1}(s) \quad (2.38)$$

where $R_m(s)$ and $P_m(s)$ are relatively right prime polynomial matrices. There exists a linear state variable feedback (l.s.r.v.) pair $\{F, G\}$, with G non singular, which satisfies the relationship:

$$\begin{aligned} T(s) &= C(sI - A - BF)^{-1}BG = R(s) P^{-1}(s) \\ &= R_m(s) P_m^{-1}(s) = T_m(s) \end{aligned} \quad (2.39)$$

if, and only if, for some non singular polynomial matrix, $H(s)$ the following three conditions hold:

- 1) $R(s) = R_m(s)H(s)$, i.e., $R(s)$ is left divisible by $R_m(s)$.
- 2) The m ordered σ_1 of $P_m(s)H(s)$ are identical to those of $P(s)$.
- 3) $P_m(s)H(s)$ is column proper.

The proof of sufficiency is constructive, and simply involves equating $P_m(s)H(s)$ to $P(s)$, once an appropriate $H(s)$ has been found, i.e., from Eq. (2.39) and substituting for $P(s)$ from Eq. (2.36)

$$P_m(s)H(s) = P(s) = (\tilde{B}G)^{-1} [S^\sigma - (A + \tilde{B}\hat{F}) S(s)] \quad (2.40)$$

Since $P_m(s)H(s)$ is column proper, the $m \times m$ matrix consisting of the coefficient of highest degree s terms in each column of $P_m(s)H(s)$ is non singular. Let this matrix be D then

$$D = (\tilde{B}G)^{-1}$$

or

$$G = (\tilde{D}B)^{-1} \quad (2.41)$$

The corresponding F can then be determined by first premultiplying Eq. (2.40) by $(\tilde{B}G)$, and then subtracting S^σ from both sides, i.e.,

$$(\tilde{B}G) P_m(s)H(s) - S^\sigma = -(\tilde{A} + \tilde{B}\hat{F})S(s) \quad (2.42)$$

Since the m ordered σ 's of $P_m(s)H(s)$ are the same (condition 2) as that of $P(s)$, in the left hand side of above Eq. (2.42), a complete cancellation of S^σ occurs, and the left hand side could be written as $-M S(s)$ where M is some constant matrix. Hence

$$-M S(s) = -(\tilde{A} + \tilde{B}\hat{F})S(s) \quad .$$

The above is satisfied if the following holds

$$\tilde{A} + \tilde{B}\hat{F} = M$$

or

$$\hat{F} = \tilde{B}^{-1}(M - \tilde{A})$$

or

$$F = \hat{F}Q = \{\tilde{B}^{-1}(M - \tilde{A})\} Q \quad (2.43)$$

Thus, the set $\{F, G\}$ is uniquely determined for a given $H(s)$, which satisfies the conditions of Theorem 2.2.

To establish necessity, consider Eq. (2.39); since $P(s)$ is non-singular

$$\begin{aligned} R(s) &= R_m(s) P_m^{-1}(s) P(s) \\ &= R_m(s) P_m^*(s) P(s) / |P_m(s)| \end{aligned} \quad (2.44)$$

where $P_m^*(s)$ is adjoint $P_m(s)$ and $|P_m(s)| = \det. P_m(s)$.

Since $R_m(s)$ and $P_m(s)$ are relatively right prime, it follows that [MacDuffee, 1956] there exists two polynomial matrices $M_1(s)$ and $M_2(s)$, such that the following relation is satisfied.

$$M_1(s)R_m(s) + M_2(s)P_m(s) = I_m \quad (2.45)$$

where I_m is the $m \times m$ identity matrix.

Postmultiplying both sides of Eq. (2.45) by $P_m^*(s) P(s)$, one obtains

$$M_1(s)R_m(s)P_m^*(s)P(s) + M_2(s)P_m(s)P_m^*(s)P(s) = P_m^*(s)P(s) \quad (2.46)$$

But from Eq. (2.44), $|P_m(s)|$ divides $R_m(s)P_m^*(s)P(s)$ because $R(s)$ is a polynomial matrix; also because

$$P_m(s)P_m^*(s) = |P_m(s)| I_m$$

$|P_m(s)|$ divides both left side members of Eq. (2.46), which implies it must also divide the right side of Eq. (2.46).

It can thus be concluded

$$P_m^{-1}(s)P(s) = \frac{P_m^*(s)P(s)}{|P_m(s)|}$$

is a nonsingular polynomial matrix, since it has been established

$|P_m(s)|$ divides $P_m^*(s)P(s)$. But from Eq. (2.44)

$$\begin{aligned} R(s) &= R_m(s) [P_m^{-1}(s)P(s)] \\ &= R_m(s) H(s) \end{aligned} \quad (2.47)$$

where $H(s) = P_m^{-1}(s)P(s)$ is a polynomial matrix. It is thus proved that $R(s)$ is left divisible by $R_m(s)$ which is condition 1 of the theorem.

Also since

$$H(s) = P_m^{-1}(s)P(s) \quad (2.48)$$

$$\Rightarrow P_m(s)H(s) = P(s) \quad (2.49)$$

Equation (2.49) directly implies conditions 2 and 3 of the theorem and, hence, the theorem is proved.

In summary, finding a pair $\{F, G\}$ for exact model matching consists of the following steps:

- (a) finding the transformed representation of the system.
- (b) factoring the desired transfer matrix (model) $T_m(s)$ as a product of two relatively prime polynomial matrices $R_m(s) P_m^{-1}(s)$.
- (c) determining the appropriate $H(s)$, if one exists.
- (d) determining the pair $\{F, G\}$ by comparing $P_m(s)H(s)$ with $P(s)$.

There are algorithms available to perform the first two of these steps. If the system is invertible, i.e., if $R(s)$ has rank m , then $H(s)$, if it exists, is uniquely given by Eq. (2.48).

However, note that just employing the above equation does not always result in a proper $H(s)$ which satisfies all three conditions of Theorem 2.2. Thus, in general, there is no algorithm to find a suitable $H(s)$, but if the system is invertible, then all one can find is whether there exists or not an $H(s)$ which satisfies all the three conditions of Theorem 2.2.

It is clear from the above discussion that the necessary and sufficient conditions for exact model matching, as given in Theorem 2.2, can be successfully employed to test whether a given system could be matched to a given model (even though the test is cumbersome). However, unlike the scalar case, the necessary and sufficient conditions fail to predict what can be matched to the system. In terms of the design no

hint is given as to what changes are required to satisfy the necessary and sufficient conditions if a design being evaluated fails to meet these conditions. The only case where these questions are completely answered, is the case of a decoupled model, which is discussed next.

2.6 The Problem of Decoupling

Necessary and sufficient conditions for decoupling a multivariable system were first given by Falb and Wolowich [1967] and were interpreted by Gilbert [1969]; The above work has been examined in detail by others as well [Slivinsky, 1969; Ferg, 1971; Agrawal, 1972], and, hence, no attempt is made to repeat it. Rather, it is assumed that the multivariable system in consideration is decouplable, i.e., necessary and sufficient conditions for decoupling are satisfied. Let the following decoupled transfer matrix be realizable and nonsingular.

$$T_d(s) = \begin{bmatrix} \frac{n_{11}(s)}{d_{11}(s)} & & \\ & \ddots & \\ & & \frac{n_{mm}(s)}{d_{mm}(s)} \end{bmatrix} \quad (2.50)$$

where $n_{ii}(s)$ is relatively prime to $d_{ii}(s)$ for $i = 1 \dots m$.

The above assumptions imply that there exists a pair $\{F, G\}$ which realizes Eq. (2.50) and the structure theorem of Section 2.4 implies that $T_d(s)$ can be written as follows:

$$T_d(s) = R(s) P^{-1}(s) \quad (2.51)$$

where $R(s)$ and $P(s)$ are as given in Eqs. (2.35) and (2.36), respectively.

Post multiply both sides of Eq. (2.51) by $P(s)$ to obtain

$$T_d(s)P(s) = R(s) \quad (2.52)$$

Now consider the i th row of both sides of Eq. (2.52) which is given by

$$\left[0 \dots 0, \frac{n_{ii}(s)}{d_{ii}(s)}, 0 \dots 0 \right] P(s) = R_i(s)$$

i.e.,

$$\frac{n_{ii}(s)}{d_{ii}(s)} P_i(s) = R_i(s) \quad (2.53)$$

where as mentioned previously in Sec. 2.2, a subscript of i on a matrix denotes i th row of the matrix.

Let us assume $p_i(s)$ and $r_i(s)$ to be the greatest common divisor of the elements in $P_i(s)$ and $R_i(s)$ respectively, and let $P'_i(s)$ and $R'_i(s)$ be the prime polynomial vectors left after taking out the greatest common divisor polynomials $p_i(s)$ and $r_i(s)$. Then the following relation is obtained.

$$\begin{aligned} P_i(s) &= p_i(s) P'_i(s) \\ R_i(s) &= r_i(s) R'_i(s) \end{aligned} \quad (2.54)$$

Substituting for $P_i(s)$ and $R_i(s)$ from above in Eq. (2.53) and multiplying both sides by $d_{ii}(s)$ the following result is obtained.

$$\{n_{ii}(s) p_i(s)\} P'_i(s) = \{d_{ii}(s) r_i(s)\} R'_i(s) \quad (2.55)$$

Notice that $\{n_{ii}(s) p_i(s)\}$ is a polynomial multiplying a prime polynomial vector $P'_i(s)$ and similarly, $\{d_{ii}(s) r_i(s)\}$ is a polynomial multiplying

prime polynomial vector $R'_i(s)$. The only way the above is possible is that

$$P'_i(s) = k R'_i(s)$$

which from Eq. (2.55) implies

$$n_{ii}(s) p_i(s) k = d_{ii}(s) r_i(s) \quad (2.56)$$

where k is a scalar nonzero constant. From above Eq. (2.56), one obtains the following:

$$\frac{k n_{ii}(s)}{d_{ii}(s)} = \frac{r_i(s)}{p_i(s)} \quad (2.57)$$

Now since $n_{ii}(s)$ is relatively prime to $d_{ii}(s)$, there must exist a polynomial $f(s)$ of a degree greater than or equal to zero, such that

$$\begin{aligned} r_i(s) &= k n_{ii}(s) f(s) \\ p_i(s) &= d_{ii}(s) f(s) \end{aligned} \quad (2.58)$$

From Eq. (2.57) and Eq. (2.58), it is clear that all the zeros of i^{th} subsystem [zeros of $n_{ii}(s)$] are contained in zeros of $r_i(s)$. In addition, there may be other zeros of $r_i(s)$ [zeros of $f(s)$] which cancel with zeros of $p_i(s)$. To show that zeros of $r_i(s)$ need not cancel with zeros of $p_i(s)$, i.e., all the zeros of $r_i(s)$ do appear as zeros of i^{th} subsystem, unless otherwise intended, it only need be shown that there exists a pair $\{F', G\}$ such that

$$\frac{r_i(s)}{p_i'(s)} = \frac{k_{ii}(s) f(s)}{d_{ii}(s) f'(s)} \quad (2.59)$$

where $p_i'(s) = d_{ii}(s) f'(s)$ is relatively prime to $r_i(s)$.

Let us choose $f'(s)$ such that it is relatively prime to $f(s)$, but the highest degree term in both polynomials is exactly the same. Thus, the coefficient of highest order term in $p'(s)$ is the same as that in $p_i(s)$, and the degree of $p'(s)$ is equal to the degree of $p_i(s)$. Now form the vector polynomial

$$P_i''(s) = p_i'(s) P_i'(s) \quad (2.60)$$

where $P_i'(s)$ is defined in Eq. (2.54), namely

$$P_i(s) = p_i(s) P_i'(s) \quad (2.54)$$

Equations (2.60) and (2.54), together with the fact that $p_i'(s)$ and $p_i(s)$ have the same degree and the same coefficient of the highest degree term, imply that $P_i''(s)$ has the same form as $P_i(s)$. Then, if $P(s)$ is column proper, the $P''(s)$ is also column proper, where

$$P''(s) = \begin{bmatrix} P_1''(s) \\ \vdots \\ P_m''(s) \end{bmatrix}$$

That is to say, if there exists a pair $\{F, G\}$ which realizes $P(s)$, then there exists a pair $\{F', G\}$ which realizes $P''(s)$.

It is thus proved that Eq. (2.59) holds, i.e., zeros of $r_i(s)$ need not cancel with the zeros of $p_i'(s)$. Since it has already been shown that zeros of i^{th} subsystem are contained in zeros of $r_i(s)$, it now implies that zeros of $r_i(s)$ are the zeros of i^{th} subsystem unless they are intentionally cancelled with the poles of i^{th} subsystem.

Since $R(s)$ is invariant, $R_i(s)$ and, hence, $r_i(s)$, is invariant. It can thus be concluded that zeros of $r_i(s)$ are invariant zeros of i^{th} subsystem, if decoupling is to be maintained. Also, all the poles of i^{th} subsystem can be arbitrarily placed because $p_i(s)$ can be made an arbitrary polynomial except for the coefficient of highest degree term without changing the form of $P(s)$ and without affecting its property of column properness.

The actual order of i^{th} subsystem is easily obtained because the degree of $p_i(s)$ equals the sum of the degree of $r_i(s)$ and the pole zero excess of the i^{th} subsystem, both of which are known.

The only other information that remains to be obtained concerns the so-called uncontrollable poles. It is well-known that if the sum of the orders of all m subsystem does not equal n , the order of the overall system, then there are some poles of the system which are unaccounted for. Gilbert [1969] and Silvisky [1969] called them uncontrollable poles because these unaccounted poles cannot be controlled by state variable feedback, if exact decoupling is to be maintained.

In practice, seldom is the model or the feedback so accurate as to be able to force the system to be exactly decoupled. Thus, these uncontrollable poles appear together with the other subsystem poles

and only in case of exact decoupled system do they cancel with system zeros at exactly the same position. Thus, if any of these uncontrollable poles are in the right half s plane, then the system is an unstable system. There is nothing one can do about these poles if decoupling is to be preserved because these poles are feedback invariant. So to ensure a stable decoupled system, one must know the positions of these so-called uncontrollable poles. It is shown next that the position of these poles is known from the knowledge of $R(s)$, the invariant part of the system transfer matrix, without even actually decoupling the system. Consider Eq. (2.51) and take the determinant of both sides

$$\begin{aligned} |T_d(s)| &= |R(s)P^{-1}(s)| = |R(s)| |P^{-1}(s)| \\ &= |R(s)| / |P(s)| \end{aligned}$$

or

$$\prod_{i=1}^m \frac{n_{ii}(s)}{d_{ii}(s)} = |R(s)| / |P(s)| \quad (2.61)$$

Now, from Eq. (2.54) $R_i(s) = r_i(s) R'(s)$, and hence,

$$R(s) = \begin{bmatrix} r_1(s) & R'_1(s) \\ \vdots & \\ r_m(s) & R'_m(s) \end{bmatrix}$$

and

$$|R(s)| = \prod_{i=1}^m r_i(s) |R'(s)|$$

where

$$R'(s) = \begin{bmatrix} R'_1(s) \\ \vdots \\ R'_m(s) \end{bmatrix} \quad (2.62)$$

Substituting the above expression for $|R(s)|$ in Eq. (2.61), one gets

$$\prod_{i=1}^m \frac{n_{ii}(s)}{d_{ii}(s)} = \frac{\prod_{i=1}^m r_i(s) |R'(s)|}{|P(s)|}$$

Substituting for $\frac{n_{ii}(s)}{d_{ii}(s)}$ from Eq. (2.57), one gets

$$\prod_{i=1}^m K_i \frac{r_i(s)}{p_i(s)} = \frac{\prod_{i=1}^m r_i(s) |R'(s)|}{|P(s)|}$$

or

$$\frac{K_1}{\prod_{i=1}^m p_i(s)} = \frac{|R'(s)|}{|P(s)|} \quad (2.63)$$

where K_i is some scalar constant.

Since the left side numerator is a nonzero constant in Eq. (2.63), the right hand side numerator polynomial $|R'(s)|$ must divide its denominator polynomial $|P(s)|$. Thus, zeros of $|R'(s)|$ must be zeros of $|P(s)|$. But zeros of $|R'(s)|$ are invariant under state variable feedback (because $R(s)$ is invariant and, hence, $R'(s)$ is invariant) and zeros of $|P(s)|$ are the overall system poles. That is to say, zeros of $|R'(s)|$ form the invariant poles of overall system. These invariant poles are the so-called uncontrollable poles of the overall decoupled system.

Thus, these uncontrollable poles are actual poles of the overall system and are cancelled by zeros in the same position if exact decoupling can be obtained. However, as mentioned earlier, in practice seldom is the model or feedback and gain so accurate as to produce exactly the decoupled system. Thus, rarely is the pole zero cancellation exact. That is to say, if any of these uncontrollable poles are in the right hand s plane, then the system is bound to be unstable if decoupled.

Since $R(s)$ is completely known $|R'(s)|$ is known and hence positions of uncontrollable pole can be found in advance without completing the actual design of the decoupled system. If any of these poles are in right hand s plane, one might as well give up the idea of decoupling the system. However, it should be remembered that the so-called uncontrollable poles are uncontrollable only as long as decoupling is required. Otherwise, if the overall system is controllable, all the n poles of the overall system can be arbitrarily placed and

thus, a completely stable system could be obtained if decoupling is not a criteria.

2.7 Summary and Conclusions

The main objective of this chapter has been to provide a comprehensive review of past and recent work on multivariable systems. Following the comments on system notation and a few definitions, Luenberger's [1967] canonical transformation is described to represent multivariable systems in phase valuable form. Next, it is shown how Wolowich and Falb [1969] used this transformation to find a structure for multivariable systems. The main advantage of such a structure is that it separates the system input output transfer matrix into state feedback invariant and state feedback dependent polynomial matrices. This property is used to find necessary and sufficient conditions for exact model matching. Lastly, the special case of decoupling is discussed and it is shown how the structure of multivariable systems in conjunction with necessary and sufficient conditions for model matching can be used to find all realizability information about decoupled models.

However, except for the decoupled models, it has been shown that the necessary and sufficient conditions for exact model matching fail to produce any realizability criteria. Thus, the necessary and sufficient conditions are useful only for testing whether a given model can be realized from the system. Even the test is cumbersome and difficult computationally, because it requires that the model transfer matrix be put into the form $T_m(s) = R_m(s) P_m^{-1}(s)$ where $R_m(s)$ and $P_m(s)$ are

relatively right prime polynomial matrices. Thus, if the test fails, then the designer has to start all over again to find some other model for which the test conditions might be satisfied. However, he has no guidance as to what changes to make and no assurance that the new model might be any more realizable than the previous one.

In the next chapter an important result is obtained which is subsequently utilized in the following chapter to give completely new and step-by-step design procedures for the design of multivariable systems.

CHAPTER 3

THE DESIGN EQUATION

3.1 Introduction and Organization of the Chapter

This and the following chapters form the main contribution of this study. The results of this chapter form the backbone for the design methods presented in the next chapter.

In Section 3.2 the concept of generalized error coefficients is extended to the multivariable case. The transfer matrix $T(s)$ and the two polynomial matrices $R(s)$ and $P(s)$, which are state variable feed-back invariant and dependent respectively, are represented in terms of their component matrices. The notation for the derivatives of these matrices is also introduced.

In Section 3.3 a result is obtained which relates the derivatives of the polynomial matrix $R(s)$ with the derivatives of the transfer matrix $T(s)$ and the polynomial matrix $P(s)$. The next section, 3.4, utilizes this result to give a relation among the component matrices of $R(s)$, $P(s)$, and $T(s)$. This relation is the main result of the chapter. Lastly, a theorem is proved which is vital to the application of the main result as described above to the design of multivariable systems. The theorem leads directly into the next chapter.

3.2 The Generalized Error Coefficients for Multivariable Systems

For the scalar case of the single input-single output system, the generalized error coefficients are directly obtained from the Maclaurin series expansion of the transfer function of the system [Truxal, 1955] as given below. Let $t(s)$ be the transfer function of a single input-single output system and $y(s)$ and $u(s)$ be the Laplace transforms of the output and input, respectively, then

$$\frac{y(s)}{u(s)} = t(s) = \frac{k_p}{1 + k_p} - \frac{1}{k_v} s - \frac{1}{k_a} s^2 \dots \quad (3.1)$$

In the classical and original definitions

k_p = position error constant

k_v = velocity error constant

k_a = acceleration error constant

The above error coefficients are measures of the steady state errors if the input is a unit step, unit ramp, unit parabolic function, and so on, respectively.

The primary disadvantage of the classical definitions rests in the limited amount of information available from the specification of error constants, since only one constant is significant. The generalized error coefficients represent an attempt to circumvent this difficulty by defining all error constants in terms of the low frequency behavior of $y(s)/r(s)$. With the Maclaurin series expansion of $t(s)$ as shown in Eq. (3.1), the error constants are defined in terms of successive coefficients in the series.

Equation (3.1) could be written as follows:

$$y(s) = \frac{k_p}{1 + k_p} u(s) - \frac{1}{k_v} s u(s) - \frac{1}{k_a} s^2 u(s) - \dots \quad (3.2)$$

If the inverse transform of series of Eq. (3.2) is taken term by term and impulses at $t=0$ are neglected, $y(t)$ is given by

$$y(t) = \frac{k_p}{1 + k_p} u(t) - \frac{1}{k_v} u'(t) - \frac{1}{k_a} u''(t) \dots \quad (3.3)$$

where dashes indicate derivatives with respect to time.

When the transfer function $t(s)$ is replaced by its Maclaurin series in Eq. (3.1), the transient terms of $y(t)$ are discarded. Thus, Eq. (3.3) is a valid description of the output $y(t)$ only after sufficient time has elapsed to allow those terms in $y(t)$ which are generated by poles of $t(s)$ to decay to insignificant amplitudes. Furthermore, the validity of Eq. (3.3) evidently depends upon the rapidity of convergence of the series of Eq. (3.1) [Savo, 1953].

Thus, only if care is exercised in the use of generalized error coefficients can it be said that the generalized error coefficients describe the relation between output and the reference input.

Equation (3.1) can be written in a more convenient manner as follows:

$$t(s) = k_1 - k_2 s - k_3 s^2 - \dots \quad (3.4)$$

Clearly,

$$k_1 = \frac{k_p}{1 + k_p} \quad ; \quad k_2 = \frac{1}{k_v} \quad , \quad k_3 = \frac{1}{k_a}$$

and so on.

In the case of multivariable system the transform of the output vector $Y(s)$ is related to the transform of input vector $U(s)$ as follows:

$$Y(s) = T(s) U(s) \quad (3.5)$$

where $T(s) = [t_{ij}(s)]$ is an $m \times m$ transfer matrix with the (ij) th entry being the transfer function $t_{ij}(s)$.

Since every entry in the transfer matrix $T(s)$ is a transfer function, a Maclaurin series expansion of the form of Eq. (2.4) could be written for each (i,j) th entry as follows:

$$t_{ij}(s) = k_{ij1} - k_{ij2}s - k_{ij3}s^2 - \dots \quad (3.6)$$

With the expansion of each term of $T(s)$ as in Eq. (3.6), $T(s)$ can be written as follows

$$T(s) = K_1 - K_2s - K_3s^2 - \dots - K_r s^{r-1} - \dots \quad (3.7)$$

where

$$K_r = [k_{ijr}]$$

is an $m \times m$ constant matrix. Thus, K_r is called the r th component of transfer matrix $T(s)$.

From Eq. (2.34), $T(s) = R(s)P(s)^{-1}$ where $R(s)$ and $P(s)$ are polynomial matrices defined in Eq. (2.35) and (2.36), respectively. Thus, $R(s)$ and $P(s)$ can also be written in the same form as that of Eq. (3.7) as follows:

$$\begin{aligned} R(s) &= R_1 + R_2s + R_3s^2 + \dots \\ P(s) &= P_1 + P_2s + P_3s^2 + \dots \end{aligned} \quad (3.8)$$

where R_i and P_i are i th component matrices of $R(s)$ and $P(s)$, respectively.

The main difference between Eq. (3.7) and (3.8) is that while Eq. (3.7) is an infinite series, Eq. (3.8) is a finite series. Let:

$$\begin{aligned} \frac{d^r}{ds^r} \{T(s)\} &\text{ be denoted by } T^r \\ \frac{d^r}{ds^r} \{R(s)\} &\text{ be denoted by } R^r \\ \frac{d}{ds^r} \{P(s)\} &\text{ be denoted by } P^r \end{aligned} \quad (3.9)$$

Then

$$P_{r+1} \triangleq \left. \frac{P^r}{r!} \right|_{s=0}$$

and

$$R_{r-1} \triangleq \left. \frac{R^r}{r!} \right|_{s=0}$$

In the next section a relation is obtained between the r^{th} derivative of $R(s)$ and the derivatives of $T(s)$ and $P(s)$.

3.3 The Derivatives of the Feedback Invariant Matrix $R(s)$

Consider Eq. (2.34) of the last chapter

$$T(s) = R(s) P^{-1}(s) \quad (2.34)$$

Post multiply both sides of Eq. (2.34) by $P(s)$ to obtain

$$R(s) = T(s)P(s) \quad (3.10)$$

In view of Eq. (3.10) and the notation of Eq. (3.9) the following result is proved next

$$R^r = \sum_{i=0}^r r_{C_i} T^{r-i} P^i \quad (3.11)$$

for $r \geq 0$

where

$$r_{C_i} = \frac{r!}{(r-i)! i!}$$

Proof: The result is proved by induction. Clearly, for the result is true for $r=0$ because for $r=0$, Eq. (3.11) becomes

$$R^0 = {}^0C_0 T^{0-0} P^0 = T^0 P^0$$

i.e.,

$$\frac{d^0}{ds^0} R(s) = \frac{d^0}{ds^0} T(s) \frac{d}{ds^0} P(s)$$

or

$$R(s) = T(s) P(s)$$

which is true from Eq. (3.10).

Let Eq. (3.11) be true for the r^{th} derivative, then expanding Eq. (3.11) for r^{th} derivative.

$$\begin{aligned} R^r &= r_{C_0} T^r P^0 + r_{C_1} T^{r-1} P^1 + \dots \\ &+ r_{C_i} T^{r-i} P^i + r_{C_{i+1}} T^{r-i-1} P^{i+1} + \dots \\ &+ r_{C_{r-1}} T^1 P^{r-1} + r_{C_r} T^0 P^r \end{aligned} \quad (3.12)$$

Differentiate the above Eq. (3.12) with respect to s to obtain the following:

$$\begin{aligned} R^{r+1} &= r_{C_0} T^{r+1} P^0 + r_{C_0} T^r P^1 + r_{C_1} T^r P^1 + r_{C_1} T^{r-1} P^2 + \dots \\ &+ r_{C_i} T^{r-i+1} P^i + r_{C_i} T^{r-i} P^{i+1} + r_{C_{i+1}} T^{r-i} P^{i+1} \\ &+ r_{C_{i+1}} T^{r-i-1} P^{i+2} + \dots + r_{C_{r-1}} T^2 P^{r-1} + r_{C_{r-1}} T^1 P^r \\ &+ r_{C_r} T^1 P^r + r_{C_r} T^0 P^{r+1} \end{aligned}$$

Combining the terms with similar derivatives in $T(s)$ and $P(s)$, one obtains

$$\begin{aligned}
 R^{r+1} &= r_{C_0} T^{r+1} P^0 + (r_{C_0} + r_{C_1}) T^r P^1 \\
 &+ \dots (r_{C_i} + r_{C_{i+1}}) T^{r-i} P^{i+1} \\
 &+ \dots (r_{C_{r-1}} + r_{C_r}) T^1 P^r + r_{C_r} T^0 P^{r+1}
 \end{aligned}
 \tag{3.13}$$

But,

$$\begin{aligned}
 r_{C_0} &= 1 = r_{C_0}^{r+1} \\
 r_{C_r} &= 1 = r_{C_{r+1}}^{r+1}
 \end{aligned}$$

and

$$\begin{aligned}
 r_{C_i} + r_{C_{i+1}} &= \frac{r!}{i!(r-i)!} + \frac{r!}{(i+1)!(r-i-1)!} \\
 &= \frac{r!}{(i+1)!(r-i)!} \{i+1 + r-i\} \\
 &= \frac{(r+1)!}{(i+1)!(r-i)!} \\
 &= r_{C_{i+1}}^{r+1}
 \end{aligned}$$

After substituting from the above identities in Eq. (3.13), the following result is obtained:

$$\begin{aligned} R^{r+1} = & r+1 C_0 + r+1 C_1 T^r P^1 + \dots + r+1 C_{i+1} T^{r-i} P^{i+1} \\ & + r+1 C_r T^1 P^r + r+1 C_{r+1} T^0 P^{r+1} \end{aligned}$$

or

$$R^{r+1} = \sum_{i=0}^{r+1} r+1 C_i T^{r+1-i} P^i \quad (3.14)$$

Comparing Eq. (3.11) with Eq. (3.14), it is evident that Eq. (3.14) is the same as Eq. (3.11) with r replaced by $r+1$.

Thus, the result of Eq. (3.11), namely

$$R^r = \sum_{i=0}^r r C_i T^{r-i} P^i \quad (3.11)$$

has been established by induction for any integer $r \geq 0$.

In the next section the central result which links component matrices of $T(s)$ with component matrices of $R(s)$ and $P(s)$ is established.

3.4 Relation Among Component Matrices

In this section a relation is obtained among the various component matrices of $T(s)$ in Eq. (3.7) and the component matrices of $R(s)$ and $P(s)$ in Eq. (3.8). This relation is examined to produce a design procedure and a theorem concerning necessary and sufficient conditions for the usefulness of the relation is proved.

After evaluating the r^{th} derivative of Eqs. (3.7) and (3.8) at $s=0$ (steady state) and after using the notation of Eq. (3.9), one gets

$$\begin{aligned} T^r \Big|_{s=0} &= \begin{cases} K_1 & \text{for } r = 0 \\ -r! K_{r+1} & \text{for } r \geq 1 \end{cases} \\ R^r \Big|_{s=0} &= \begin{cases} R_1 & \text{for } r=0 \\ r! R_{r+1} & \text{for } r \geq 1 \end{cases} \\ P^r \Big|_{s=0} &= \begin{cases} P_1 & \text{for } r = 0 \\ r! P_{r+1} & \text{for } r \geq 1 \end{cases} \end{aligned} \quad (3.15)$$

Now, evaluate Eq. (3.11) at $s=0$ and substitute from Eq. (3.15) to obtain

$$R^r \Big|_{s=0} = \sum_{i=0}^r r_{C_i} T^{r-i} \Big|_{s=0} P^i \Big|_{s=0}$$

Now, evaluate the above equation for $r=0$ to obtain

$$R_1 = K_1 P_1 \quad (3.16)$$

and evaluate for $r \geq 1$ to obtain the following result.

$$R^r \Big|_{s=0} = r_{C_r} T^0 P^r \Big|_{s=0} + \sum_{i=0}^{r-1} r_{C_i} T^{r-i} \Big|_{s=0} P^i \Big|_{s=0} \quad \text{for } r \geq 1$$

After substituting for the derivatives from Eq. (3.15), one obtains

$$r!R_{r+1} = r!K_1P_{r+1} + \sum_{i=0}^{r-1} \frac{r!}{i!(r-i)!} \{-(r-i)! K_{r+1-i}\} i! P_{i+1}$$

i.e.,

$$r!R_{r+1} = r!K_1P_{r+1} - \sum_{i=0}^{r-1} r! K_{r+1-i} P_{i+1} \quad \text{for } r \geq 1$$

Cancelling the common non zero coefficient $r!$ from both sides of the above equation and taking the summation term on the left, the result is

$$K_1P_{r+1} = R_{r+1} + \sum_{i=0}^{r-1} K_{r+1-i} P_{i+1} \quad \text{for } r \geq 1$$

The above can be written as follows

$$K_1P_{r+1} = R_{r+1} + \sum_{i=1}^{r-1} K_{r+1-i} P_{i+1} + K_{r+1} P_1 \quad (3.17)$$

Break Eq. (3.17) in two parts, one for $r=1$ and the other for $r \geq 2$, and write together with Eq. (3.16) as follows:

$$\begin{aligned}
&\text{for } r=0 \quad K_1 P_1 = R_1 \\
&\text{for } r=1 \quad K_1 P_2 = R_2 + K_2 P_1 \\
&\text{for } r \geq 2 \quad K_1 P_{r+1} = R_{r+1} + \sum_{i=1}^{r-1} K_{r+1-i} P_{i+1} \\
&\qquad\qquad\qquad + K_{r+1} P_1
\end{aligned} \tag{3.18}$$

The set of Eqs. (3.18) could be written as one single equation for all value of $r \geq 0$ as follows:

$$K_1 P_{r+1} = Q_r + K_{r+1} P_1 \tag{3.19}$$

where

$$Q_r = \begin{cases} R_1 - K_1 P_1 & \text{for } r = 0 \\ R_2 & \text{for } r = 1 \\ R_{r+1} + \sum_{i=1}^{r-1} K_{r+1-i} P_{i+1} & \text{for } r \geq 2 \end{cases}$$

Evaluating Eq. (3.19) for $r = 0, 1, 2, 3 \dots$ in sequence, the following sequence of equations is obtained:

$$\begin{aligned}
K_1 P_1 &= R_1 \\
K_1 P_2 &= R_2 + K_2 P_1 \\
K_1 P_3 &= R_3 + K_2 P_2 + K_3 P_1 \\
&\vdots
\end{aligned} \tag{3.20}$$

From the sequence of Eqs. (3.2), it is evident that $P_1, P_2, P_3 \dots$ can be calculated in sequence for any given $K_1, K_2, K_3 \dots$ respectively, if K_1 is nonsingular. Thus, nonsingularity of K_1 is a sufficient condition for unique solution to the components of $P(s)$. The following theorem is presented here to prove the necessity of K_1 being nonsingular for any useful design.

Theorem 3.1: K_1, P_1 , and R_1 all must be nonsingular for any stable design for which transfer matrix $T(s)$ is nonsingular.

Note that the reverse is not true; i.e., the system could be unstable and still have nonsingular K_1, P_1 and R_1 . However, the theorem tells that if any of K_1, P_1 , or R_1 is singular, then either the design is unstable or the transfer matrix obtained is singular.

Proof of Theorem 3.1: From Eq. (2.36) $P(s)$ can be calculated for any given $\{F, G\}$. Also, from Eq. (3.8)

$$P_1 = P(s) \Big|_{s=0}$$

Then

$$|P_1| \stackrel{\Delta}{=} |P(s)| \Big|_{s=0}$$

Let

$$|P(s)| = (s + \lambda_1)(s + \lambda_2) \dots (s + \lambda_n)$$

Then

$$|P(s)| \Big|_{s=0} = \lambda_1 \lambda_2 \cdot \cdot \cdot \lambda_n$$

or

$$|P_1| = \lambda_1 \lambda_2 \cdot \cdot \cdot \lambda_n$$

Now, if the system design is a stable design, then none of the eigenvalues is zero and, hence, $|P_1| \neq 0$; i.e., P_1 is nonsingular. Note once again that the above does not imply that λ 's are all negative or have real part negative. All it implies is that if P_1 is singular, then one of the eigenvalues is zero, resulting in an unstable design.

Now, from Eq. (2.34)

$$T(s) = R(s) P^{-1}(s)$$

or

$$P(s) = T^{-1}(s) R(s)$$

since $T(s)$ is assumed to be nonsingular.

From the above, one easily obtains

$$P(s) \Big|_{s=0} = T^{-1}(s) \Big|_{s=0} R(s) \Big|_{s=0}$$

Let

$$T_1 = T^{-1}(s) \Big|_{s=0}$$

then from above, it follows that

$$P_1 = T_1 R_1$$

But the above equation and the fact that P_1 must be nonsingular implies R_1 must also be nonsingular. However, evaluating Eq. (2.34) at $s=0$ gives

$$T(s) \Big|_{s=0} = R(s) \Big|_{s=0} P^{-1}(s) \Big|_{s=0}$$

$$\text{i.e., } K_1 = R_1 P_1^{-1}.$$

The above is valid because P_1 is nonsingular; i.e., P_1^{-1} exists. Now, since both matrices on the right are nonsingular, it implies K_1 must also be nonsingular. Hence, the theorem is proved.

3.5 Summary

The information developed in this chapter is new and forms the basis for the design of the multivariable systems as discussed in detail in the next chapter. The main result obtained in this chapter is the identity of Eq. (3.19) from which a step-by-step design procedure can be developed. The sequential nature of design procedures to be developed is indicated by Eq. (3.20), where it is seen that each component of $P(s)$ can be determined from the knowledge of the corresponding component of $T(s)$ and previous components of $P(s)$.

In the last section a theorem (Theorem 3.1) is presented which established usefulness of Eq. (3.19). If existence of K_1 inverse were not necessary for a stable design with the nonsingular transfer matrix, then one would have put too much of a restriction on K_1 by saying that

K_1 must be nonsingular. Ideally, one would like to see all elements in K_1 to be unity because (ij) th element of K_1 is nothing but the value of

$$y_i(s)/u_j(s) \Big|_{s=0}$$

i.e., steady state response to a unit step input. But, by Theorem 3.1, it is seen that if K_1 is singular, then either at least one of the system pole lies at origin or the transfer matrix is singular (the output responses are not independent) both of which are undesirable situations. Thus, Theorem 3.1 gives significant information about what steady state response can be achieved for multivariable systems.

CHAPTER 4

A DESIGN METHOD FOR MULTIVARIABLE SYSTEMS

4.1 Introduction and Organization of the Chapter

In this chapter the results of the previous chapter are utilized to develop a design method and to indicate constraints on design requirements. First, additional notation is introduced in Sec. 4.2 to compactly represent columns, rows and elements of the component matrices. Next in Sec. 4.3 a theorem concerning the design constraints is proved. The proof of the theorem is constructive. A step-by-step design procedure is outlined in Sec. 4.4. The design procedure is illustrated with the help of a simple example in Sec. 4.5. In the last section the results of the chapter are summarized and the advantages and drawbacks of the design method are discussed. It is pointed out how some of the unwanted features of this design method could be overcome. This is done in the next chapter.

4.2 Notation and Identity

Unless specifically mentioned, the following notation is used throughout.

1. A polynomial matrix $P(s)$ is represented in the following equivalent manners.

$$P(s) = [p_{ij}(s)] = P_1 + P_2 s + P_3 s^2 + \dots + P_{r+1} s^r + \dots$$

so that $p_{ij}(s)$ denotes (ij) th element of the matrix $P(s)$

and

$$P_{r+1} = \frac{1}{r!} \left. \frac{d^r}{ds^r} P(s) \right|_{s=0}$$

is a component matrix of $P(s)$ for $r = 0, 1, 2 \dots$

2. A $k \times l$ submatrix of a component matrix P_r is denoted by specifying all its elements inside a $k \times l$ constant matrix followed by a subscript r as shown below:

$$\begin{bmatrix} p_{ij} & p_{ij+1} & \dots & p_{ij+l} \\ p_{i+1j} & & & \\ \vdots & & & \\ p_{i+kj} & & \dots & p_{i+kj+l} \end{bmatrix}_r$$

Also $(p_{ij})_r$ denotes (ij) th element of P_r .

3. The i th column of a component matrix P_r is denoted by the following equivalent notations:

$$p_r^i = [p^i]_r$$

Thus, a group of columns of P_r can be denoted in the following equivalent ways:

$$[p_r^i \ p_r^{i+1} \ \dots \ p_r^j] = [p^i \ p^{i+1} \ \dots \ p^j]_r$$

Clearly, the above is a matrix with $j-i+1$ columns in it.

Accordingly, P_r can be denoted in terms of its columns as follows:

$$P_r = [p_r^1 \ p_r^2 \ \dots \ p_r^m] = [p^1 \ p^2 \ \dots \ p^m]_r$$

4. In an earlier section, a subscript is used to denote the row of a matrix. Here, however, subscripts are used for components of polynomial matrices. Thus, to avoid confusion and to be consistent with an earlier notation, the component matrix or the submatrix of a component matrix is first denoted by a single capital letter. The rows are then denoted by the usual notation, i.e., by a subscript. For example, to denote the rows of a component matrix, p_j , let $W = P_j$. Then w_i denotes the i th row of the matrix P_j . Thus, the notation introduced here is compatible with the earlier notation where a subscript i denotes i th row and a superscript i denotes i th column of a constant matrix.

Next, an identity is established which is useful in proving the theorem of the next section. The identity essentially partitions the multiplication of two matrices in such a way that the known part is separated from the unknown part.

Identity 4.1: Let $VW = Z$ where V , W , and Z are constant matrices of dimension $m \times p$, $p \times q$ and $m \times q$, respectively (m , p , and q are integers). Let α_1 th, α_2 th ... α_p th denote the p ordered columns of V and the same p ordered rows of W . If $j \leq m$, then the product $Z = VW$ can be broken into two matrices as shown below.

$$Z = [v^{\alpha_1} \ v^{\alpha_2} \ \dots \ v^{\alpha_j}] \begin{bmatrix} w_{\alpha_1} \\ w_{\alpha_2} \\ \vdots \\ w_{\alpha_j} \end{bmatrix} + [v^{\alpha_{j+1}} \ v^{\alpha_{j+2}} \ \dots \ v^{\alpha_m}] \begin{bmatrix} w_{\alpha_{j+1}} \\ w_{\alpha_{j+2}} \\ \vdots \\ w_{\alpha_m} \end{bmatrix} \quad (4.1)$$

That is to say, Z is the sum of two matrices, the first of which is made from j ordered columns of V and corresponding j ordered rows of W . The second matrix is the multiplication of the remaining $(p-j)$ ordered columns of V and the corresponding $(p-j)$ ordered rows of W . The above in effect is a way of partitioning the multiplication of two matrices into two parts.

Proof of the Identity: Clearly

$$Z = VW = [v^1 \ v^2 \ \dots \ v^p] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{bmatrix} \quad (4.2)$$

where, as stated above, v^i denotes i th column of V and w_i denotes i th row of W . The above Eq. (4.2) can be rewritten as follows:

$$v^1 w_1 + v^2 w_2 + v^3 w_3 + \dots + v^p w_p = Z$$

Thus, Z is the sum of p matrices each of dimension $(m \times q)$ as shown above. By reordering, the above can be written in the following equivalent manner.

$$Z = \sum_{i=1}^p v^i w_i = v^{\alpha 1} w_{\alpha 1} + v^{\alpha 2} w_{\alpha 2} + \dots + v^{\alpha p} w_{\alpha p} \quad (4.3)$$

From Eq. (4.3), by combining the p reordered columns of v and the p ordered rows of W , one gets:

$$Z = [v^{\alpha 1} \ v^{\alpha 2} \ \dots \ v^{\alpha p}] \begin{bmatrix} w_{\alpha 1} \\ w_{\alpha 2} \\ \vdots \\ w_{\alpha m} \end{bmatrix} \quad (4.4)$$

From Eq. (4.4) it is clear that if the p columns of V are reordered and the p rows of W are reordered accordingly, then the result of the multiplication is unchanged.

Now, partition Eq. (4.3) as follows:

$$\begin{aligned}
 Z = & v^{\alpha_1} w_{\alpha_1} + v^{\alpha_2} w_{\alpha_2} + \dots + v^{\alpha_j} w_{\alpha_j} \\
 & + v^{\alpha_{j+1}} w_{\alpha_{j+1}} + v^{\alpha_{j+2}} w_{\alpha_{j+2}} + \dots + v^{\alpha_p} w_{\alpha_p}
 \end{aligned}$$

(4.5)

In the above equation combine the first j matrices in one batch and the remaining $(p-j)$ matrices in the second batch to obtain the required identity of Eq. (4.1). The identity is thus established. This identity is used in the proof of Theorem 4.1.

4.3 Design Method I

The design described in this section is based mainly upon the result derived in the last chapter, especially as expressed in Eq. (3.19). It is seen from the form of Eq. (3.20) that $P_1, P_2, P_3 \dots$ can all be determined in sequence from the specification of $K_1, K_2, K_3 \dots$, respectively. It is also seen that the generalized error coefficients describe relations between the output and the reference input. However, it is unfortunate that in the general case the P_i 's must satisfy additional constraints, and thus the K_i 's are correspondingly restricted. The constraints on $P(s)$ are briefly described in Section 2.4 and 2.5.

To begin the development, Eq. (2.36) is now examined in view of necessary and sufficient conditions for exact model matching as given in Section 2.5. Recall that in Section 2.5, $P(s)$ is given as

$$\begin{aligned}
P(s) &= (\tilde{B}G)^{-1} \{s^\sigma - (\tilde{A} + \tilde{B}\hat{F}) S(s)\} \\
&= (\tilde{B}G)^{-1} s^\sigma - (\tilde{B}G)^{-1} (\tilde{A} + \tilde{B}\hat{F}) S(s)
\end{aligned} \tag{2.36}$$

Irrespective of what \tilde{B} and G are, if $(\tilde{B}G)^{-1}$ exists, then any polynomial in the i th column of $P(s)$ has a maximum degree of σ_i . This is true because any polynomial in the i th column of $(\tilde{B}G)^{-1}(\tilde{A} + \tilde{B}\hat{F}) S(s)$ has a maximum degree of $\sigma_i - 1$, and any polynomial in the i th column of $(\tilde{B}G)^{-1} s^\sigma$ has a maximum degree of σ_i . However, $(\tilde{B}G)^{-1}$ must exist for realizability, since the condition that $P(s)$ be column proper for model matching comes from the existence of $(\tilde{B}G)^{-1}$.

Thus, for realizability the degree of the highest order polynomial in each column of $P(s)$ must remain unchanged as this has been shown to be one of the necessary and sufficient condition for model matching in Sec. 2.5. Also $P(s)$ must be column proper for a nonsingular transfer function (Sec. 2.5). By definition $P(s)$ is column proper if the matrix consisting of the coefficients of the highest degree term in each column is nonsingular, i.e., $(\tilde{B}G)^{-1}$ is nonsingular. But it has just been shown that the highest degree of any polynomial in the i th column of $P(s)$ is σ_i . Also from Eq. (3.8), $P(s)$ is given as follows

$$P(s) = P_1 + P_2 s + P_3 s^2 + \dots$$

Thus, from the above representation of $P(s)$, it is clear that the column vector consisting of coefficient of s^{σ_i} in the i th column of $P(s)$ is

nothing but the i th column of the component matrix P_{σ_i+1} which is represented by $p_{\sigma_i+1}^i$. Therefore, $P(s)$ is column proper if the $m \times m$ matrix

$$(\tilde{B}G)^{-1} = \begin{bmatrix} p_{\sigma_1+1}^1 & p_{\sigma_2+1}^2 & \cdots & p_{\sigma_m+1}^m \end{bmatrix} \quad (4.6)$$

is nonsingular.

Since column properness of $P(s)$ is one of the necessary and sufficient conditions for model matching. (See Sec. 2.5), in summary the constraints on the form of $P(s)$ can be stated as follows:

- (1) $p_{\sigma_i+1}^i$ must be an independent vector for all $1 \leq i \leq m$
 - (2) $p_{\sigma_i+j}^i = 0$ for $j \geq 2$ and for all $1 \leq i \leq m$
- (4.7)

The above two constraints in turn limit the realizable set of K_i 's and require certain sufficiency conditions to be satisfied. These conditions are taken into account by Theorem (4.1). The following definition is introduced to aid in a precise statement of that theorem.

Definition 4.1 Component definite polynomial matrix: Let

- (1) the polynomial matrix $P(s)$ be represented by Eq. (3.8), i.e.,

$$P(s) = P_1 + P_2 s + P_3 s^2 + \dots \quad (3.8)$$

- (2) γ_i be the number of zero columns in P_i

- (3) Γ_i be a $\gamma_i \times \gamma_i$ constant matrix formed by the elements of P_1 at the cross section of the γ_i ordered rows and the same γ_i ordered columns, which correspond to γ_i zero columns of P_i .

Then $P(s)$ is called "component definite" if Γ_i are nonsingular for all $i \geq 1$.

Notice that to test for component definiteness of $P(s)$, one need not form Γ_i for $i = 1$ and 2. This is true because each of the σ 's is greater than or equal to one; therefore, P_1 and P_2 do not have any zero column. Also, since $P(s)$ consists of finite degree polynomials only, there exists a finite integer $\beta \leq n$, the order of multivariable system under discussion, such that $P_\beta = 0$ and thus $\Gamma_\beta = P_1$. Hence, $P(s)$ is component definite only if P_1 is nonsingular. The nonsingularity of P_1 is established in Theorem 3.1 and, hence, does not cause any new constraint.

Theorem 4.1 is concerned with the freedom of choice of the transfer function elements of the overall transfer matrix $T(s)$. This freedom of choice is expressed in terms of the freedom of choice for the error coefficients.

Theorem 4.1 The first $\sigma_i + 1$ error coefficients of each transfer function element $t_{ji}(s)$ in i th column of $T(s)$ can be realized arbitrarily by state variable feedback alone if 1) the first error coefficients k_1^i are chosen such that $P(s)$ is forced to be component definite; 2) the

last error coefficients $k_{\sigma_i+1}^i$ are chosen such that resulting $p_{\sigma_i+1}^i$ are mutually independent vectors for all $i \leq m$.

The above theorem gives only sufficient conditions for the freedom of choice of the error coefficients of each transfer function. The application of the theorem for design of multivariable systems puts constraints on only first and last generalized error coefficient for each transfer function. The first of the two conditions imposed by the theorem is that K_1 be chosen such that the resulting $P(s)$ is component definite. The test for component definiteness of $P(s)$ does not require lengthy computation because the form of $P(s)$ is completely known in advance from the knowledge of the σ_i 's, and the matrix P_1 is completely determined from the knowledge of the matrices K_1 and R_1 . Thus, one can easily form the $\gamma_i \times \gamma_i$ matrices Γ_i (Def. 4.1) and test them for nonsingularity. If some Γ_i happens to be singular for the particular choice of the matrix K_1 , the K_1 must be changed accordingly. The second condition of the theorem restricts the last specifiable error coefficient to produce a $P(s)$ which is column proper and thus realizable.

The proof is accomplished by first assuming that the conditions of the theorem are satisfied, and then showing that all the components of $P(s)$ are uniquely determined and that they form a $P(s)$ which is realizable by the state variable feedback of Eq. (2.2).

4.3.1 Proof of Theorem 4.1

Examine the first equation of the series of Eq. (3.20), namely

$$K_1 P_1 = R_1$$

From Theorem (3.1) it is seen that K_1 and P_1 must be nonsingular for any stable system whose transfer matrix is nonsingular. Thus, knowing R_1 from the knowledge of $R(s)$ and K_1 as specified, P_1 is readily given as

$$P_1 = K_1^{-1} R_1$$

It is assumed that the P_1 as obtained above is such as to force $P(s)$ to be component definite. If not, one must change K_1 to force $P(s)$ to be component definite.

Now, examine Eq. (3.19) as given below

$$K_1 P_{r+1} = Q_r + K_{r+1} P_1 \quad (3.19)$$

where, as denoted also in Eq. (3.19)

$$Q_r = R_{r+1} \quad \text{for } r = 1$$

$$Q_r = R_{r+1} + \sum_{i=1}^{r-1} K_{r+1-i} P_{i+1} \quad \text{for } r \geq 2$$

For convenience, let $j = r + 1$, then Eq. (3.19) can be written as follows:

$$K_1 P_j = Q_r + K_j P_1 \quad (4.8)$$

Thus in Eq. (4.8), Q_r is completely known at the time P_j is being evaluated, provided P_j is evaluated in strict sequence for $j = 2, 3, 4, \dots$. This sequential evaluation of the matrices P_i is a significant computational advantage.

Now, examine the form of $P(s)$. Since the degree of the highest degree term in the i th column of $P(s)$ is σ_i , the i th column of P_j is identically zero for $j > \sigma_i + 1$. Thus, in general, P_j might have some columns identically zero. Let P_j have γ_j zero columns, then one need only to show that the remaining $m - \gamma_j$ columns of the matrix K_j are still arbitrarily specifiable, because the theorem promises only $\sigma_i + 1$ arbitrary error coefficients for each element in the i th column of the transfer matrix $T(s)$.

To prove the above, it suffices to show that if $\ell = \gamma_j$ columns of P_j are arbitrarily specified, then in Eq. (4.8), the corresponding ℓ columns of K_j are uniquely determined from the knowledge of remaining $m - \ell$ columns of K_j . P_j can then be determined completely from the same Eq. (4.8). (Notice that if ℓ columns of P_j can be specified arbitrarily, they could be specified as zero.) The above is proved next.

Let α_1 th, α_2 th, \dots α_ℓ th columns of P_j be specified arbitrarily and let θ_1 th, θ_2 th, \dots $\theta_{m-\ell}$ th columns of P_j be the remaining unspecified columns to be determined. Then by considering the ℓ specified columns, namely α_1 th, α_2 th, \dots α_ℓ th of Eq. (4.8), the following relationship is obtained

$$K_1 [p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_\ell}]_j = [q^{\alpha_1} q^{\alpha_2} \dots q^{\alpha_\ell}]_r + K_j [p^{\alpha_1} p^{\alpha_2} \dots p^{\alpha_\ell}]_1$$

The above could be written as follows:

$$K_j W = Z$$

where

$$W = [p^{\alpha 1} \ p^{\alpha 2} \ \dots \ p^{\alpha \ell}]_1$$

is a $(m \times \ell)$ matrix and

$$Z = K_1 [p^{\alpha 1} \ p^{\alpha 2} \ \dots \ p^{\alpha \ell}]_j - [q^{\alpha 1} \ q^{\alpha 2} \ \dots \ q^{\alpha \ell}]_r$$

is a $(m \times \ell)$ matrix.

Since $p_j^{\alpha i}$ are specified and Q_r and K_1 are completely known in advance, Z is completely known. By expanding the left side of the relation $K_j W = Z$, it could be written as follows:

$$[k^1 \ k^2 \ \dots \ k^m]_j \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = Z$$

where w_i is $(1 \times \ell)$ row vector.

Applying the identity of Eq. (4.5), the left side of the above equation could be separated into two parts, one containing the ℓ unknown columns of K_j and the other containing $m-\ell$ arbitrarily specified columns of K_j , as follows:

$$[k^{\alpha_1} k^{\alpha_2} \dots k^{\alpha_\ell}]_j \begin{bmatrix} w_{\alpha_1} \\ w_{\alpha_2} \\ \vdots \\ w_{\alpha_\ell} \end{bmatrix} + [k^{\theta_1} k^{\theta_2} \dots k^{\theta_{m-\ell}}] \begin{bmatrix} w_{\theta_1} \\ w_{\theta_2} \\ \vdots \\ w_{\theta_{m-\ell}} \end{bmatrix} = Z \quad (4.10)$$

Now, examine the following $\ell \times \ell$ matrix

$$\begin{bmatrix} w_{\alpha_1} \\ w_{\alpha_2} \\ \vdots \\ w_{\alpha_\ell} \end{bmatrix}$$

Notice that the matrix W itself consists of ℓ columns of P_1 corresponding to zero columns of P_j , and w_{α_i} is the α_i the row of W . Thus, by the definition of component definiteness of $P(s)$, the matrix under consideration is nothing but the matrix Γ_j formed while testing $P(s)$ for a component definiteness. Therefore, Eq. (4.10) can be written as follows:

$$[k^{\alpha_1} k^{\alpha_2} \dots k^{\alpha_\ell}]_j \Gamma_j = Z - [k^{\theta_1} k^{\theta_2} \dots k^{\theta_{m-\ell}}] \begin{bmatrix} w_{\theta_1} \\ w_{\theta_2} \\ \vdots \\ w_{\theta_{m-\ell}} \end{bmatrix} \quad (4.11)$$

Since Γ_j is an $\ell \times \ell$ (here $\ell = \gamma_j$) nonsingular matrix for all j , the ℓ unspecified columns of K_j on the left side of Eq. (4.11) can be determined by multiplying on both sides of Eq. (4.11) by $(\Gamma_j)^{-1}$.

The matrix K_j is thus completely known, and hence P_j can be completely calculated using Eq. (4.8), since everything except P_j is known in that equation and K_1 is a nonsingular matrix. The matrix P_j so obtained is guaranteed to have ℓ columns as specified (identically zero in this particular case).

Proceeding sequentially for $j = 2, 3$, all the components of $P(s)$ can be calculated easily. The $P(s)$ thus obtained is guaranteed to have required form (the degree of the highest degree term in the i th column of $P(s)$ is σ_i). The only remaining requirement for $P(s)$ to be realizable is that it be column proper, i.e., $p_{\sigma_i+1}^i$ be an independent vector for $i = 1, 2 \dots m$. However, for some particular choice of $k_{\sigma_i+1}^i$, $p_{\sigma_i+1}^i$ may not come out to be an independent vector. In that case, one must change $k_{\sigma_i+1}^i$ to be an independent vector.

To see that $p_{\sigma_i+1}^i$ can be made independent by the proper choice of $k_{\sigma_i+1}^i$, it suffices to prove that $k_{\sigma_i+1}^i$ can be found for any arbitrary choice of $p_{\sigma_i+1}^i$. But this follows from the fact that $P(s)$ is component definite. Thus, if while solving for $p_{\sigma_i+1}^i$ the columns $p_{\sigma_i+1}^i$ come out to be dependent, then choose them as desired and solve Eq. (4.8) as before, except that this time $\ell = \gamma_{j+1}$. Thus, proceeding as before, Eq. (4.8) can now be solved for $\ell = \gamma_{j+1}$ unknown columns of K_j instead of γ_j columns of K_j . The solution is once again guaranteed

because Γ_{j+1} is also nonsingular by the definition of component definiteness of $P(s)$. The theorem is thus proved.

Notice that the constraint that $P(s)$ be component definite is easy to handle because this simply requires that P_1 satisfy certain properties. However, $P_1 = k_1^{-1} R_1$ and is calculated at the very beginning of design. Hence, it is easy to alter. Also, the constraint that $k_{\sigma_i+1}^i$ be chosen so as to force each $p_{\sigma_i+1}^i$ to be an independent vector follows from the column properness of $P(s)$. Thus, this constraint amounts to choosing the last error coefficient for each transfer function in such a way that the matrix of Eq. (4.6) as given below

$$\begin{bmatrix} p_{\sigma_1+1}^1 & p_{\sigma_1+1}^2 & \cdots & p_{\sigma_m+1}^m \end{bmatrix}$$

is a nonsingular matrix. This requires relatively small changes in the error coefficients $k_{\sigma_i+1}^i$.

In the next section the steps involved in applying the design method of this chapter are summarized.

4.4 Step-by-Step Design Procedure

The following step-by-step design procedure simplifies the presentation of the computational techniques involved in designing the multivariable system by using the design method of the chapter.

Step 1: Find the transformation matrix Q as follows:

- (a) Find the first n independent columns of the controllability matrix.

(b) Rearrange these n independent columns to form the lexographic matrix and find σ 's from there.

(c) Take the inverse of this lexographic matrix and form the transformation matrix Q as explained in Sec. 2.3.

Step 2: Apply the transformation to find the new state variable representation of the system; that is, find \hat{A} , \hat{B} , and \hat{C} .

Step 3: Use the results of Step 1 and Step 2 to form the matrices S^{σ} , $S(s)$, \tilde{A} and \tilde{B} .

Step 4: Calculate the matrix $R(s)$ and the most general form of the matrix $P(s)$.

If a decoupled system is desired, follow Steps (5d) through (9d); otherwise follow Steps (5) through (12).

Step 5d: Test if the system is decouplable and find the pole zero excess of each subsystem during this test.

Step 6d: Find the fixed zeros for each subsystem from the knowledge of the matrix $R(s)$.

Step 7d: Determine the position of uncontrollable poles, if any, by examining $R(s)$.

Step 8d: Choose the model transfer matrix for the system which has the fixed zeros as found in Step (6d) and pole zero excess as found in Step (5d).

Step 9d: Find the pair $\{F, G\}$ to realize the above decoupled transfer matrix.

For the general case of model matching, follow the steps listed below:

- Step 5: Expand each $t_{ij}(s)$ of the model transfer matrix in Maclaurin series up to σ_i+1 terms. The specifiable portion of k_i 's is then completely known.
- Step 6: Write $R(s)$ in its component form. Since $R(s)$ is completely known from Step (4) all components of $R(s)$ are known.
- Step 7: Calculate P_1 from the knowledge of R_1 and K_1 .
- Step 8: Test $P(s)$ for component definiteness. To do this, form the Γ_i 's from the knowledge of the form of $P(s)$ and P_1 . If all of them are nonsingular, proceed to Step (9). Otherwise change k_1 accordingly and go back to Step (7).
- Step 9: Calculate P_2, P_3, \dots in sequence until all the components of $P(s)$ are known. At the end of each sequence, check if the $p_{\sigma_i+1}^i$ are independent vectors. If not, change the $k_{\sigma_i+1}^i$ accordingly and repeat the last sequence.
- Step 10: Find the transfer matrix $T(s) = R(s) P^{-1}(s)$.
- Step 11: Find the pair $\{\hat{F}, G\}$ to realize the $P(s)$ obtained above.
- Step 12: Using the transformation matrix Q , find the pair $\{F, G\}$ which realizes the transfer matrix of Step (10).

The step-by-step design procedure is illustrated with a simple example in the next section.

4.5 Example 4.1

The example here has been intentionally chosen to be simple in order to meaningfully and concisely illustrate all the steps involved in

the design of multivariable systems using generalized error coefficients. The example is a slightly changed version of Example 3.1 given in Silvinsky's dissertation [1969]. The problem of decoupling is illustrated first and the problem of complete model matching is tackled next.

Consider the multivariable system whose block diagram is given in Fig. 4.1 and which is described by the following state equations.

$$\begin{aligned} \dot{X} &= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} U \\ Y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} X \end{aligned} \quad (4.11)$$

Obviously

$n = \text{order of system} = 3$

$m = \text{number of inputs} = \text{number of outputs} = 2$

$X \in E^3$ is a column vector

Y and $U \in E^2$ are column vectors

By inspection, both the input matrix B and output matrix C of Eq. (2.1) have full rank, $m=2$. Also, system is completely controllable and completely observable.

Step 1: The controllability matrix of Eq. (2.7) is obtained as follows:

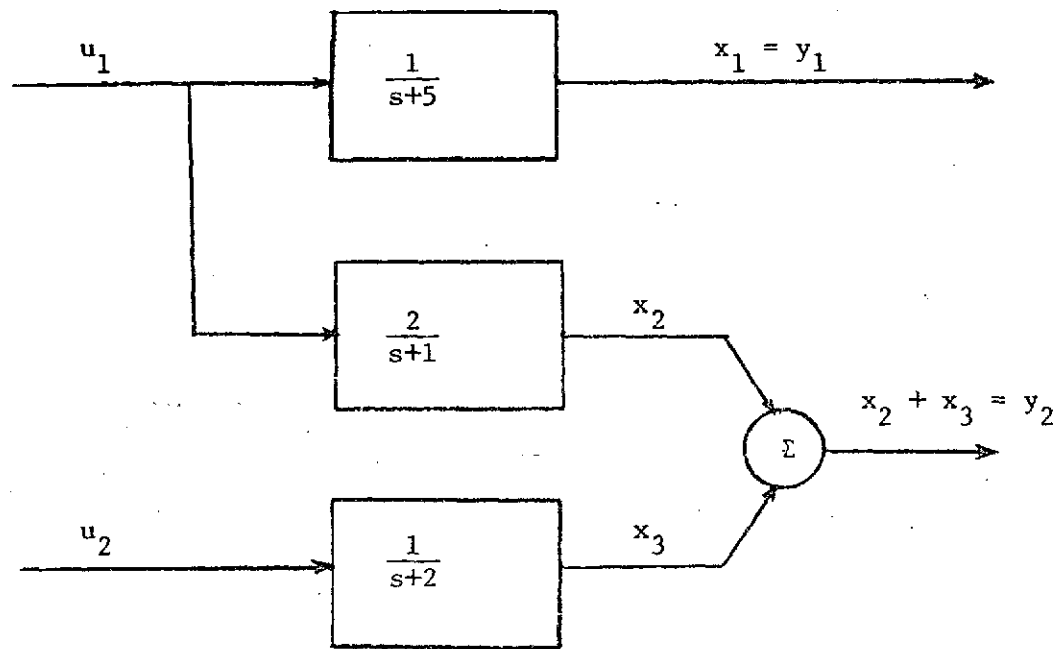


Fig. 4.1 Block Diagram of the System

$$[B \ AB \ A^2B] = \begin{bmatrix} 1 & 0 & -5 & 0 & 25 & 0 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 4 \end{bmatrix} \quad (4.12)$$

Clearly, the first n independent columns of the controllability matrix are the first three columns of the matrix of Eq. (4.12).

After rearranging these first three independent columns according to Eq. (2.8), one obtains

$$[b^1 \ Ab^1 \ b^2] = [b^1 \ A^{\sigma_1-1}b^1 \ A^{\sigma_2-1}b^2] \quad (4.13)$$

Accordingly, $\sigma_1 = 2$ and $\sigma_2 = 1$. Notice that for this particular example, the σ 's are unique because the vectors b^2 , Ab^2 , and A^2b^2 are dependent vectors. Hence, $\sigma_2 = 1$, leaving no choice for σ_1 except that $\sigma_1 = 2$ as stated above.

The inverse of the matrix in Eq. (4.13) is

$$\begin{bmatrix} 1 & -5 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.25 & 0.625 & 0 \\ -0.25 & 0.125 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \quad (4.14)$$

From Eq. (2.9), $d_1 = \sigma_1 = 2$, $d_2 = \sigma_1 + \sigma_2 = 3$ and, hence, ℓ_1 and ℓ_2 are defined as the second and third row, respectively, of the matrix in Eq. (4.14).

Hence, from Eq. (2.10), the transformation matrix is given by

$$Q = \begin{bmatrix} \lambda_1 \\ \lambda_1 A \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -0.25 & 0.125 & 0 \\ 1.25 & -0.125 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \quad (4.15)$$

Step 2: Using the transformation $\hat{X} = QX$, the transformed system of Eq.

(2.11) is found, where the transformed matrices are defined by Eq.

(2.12). Thus, the transformed system is given by

$$\dot{\hat{X}} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -5 & -6 & 0 \\ \hline 0 & 0 & -2 \end{array} \right] \hat{X} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right] U \quad (4.16)$$

$$Y = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 10 & 2 & 1 \end{array} \right] \hat{X} \quad (4.17)$$

As can be checked by inspection, the transformed system matrices in Eq. (4.16) do have the special form described in Eq. (2.15), (2.16), and (2.17).

Step 3: The matrices S^σ , $S(s)$, \tilde{A} and \tilde{B} , are found using Equations (2.18) through (2.21), respectively, as given below:

$$S^\sigma = \begin{bmatrix} s^{\sigma 1} & 0 \\ 0 & s^{\sigma 2} \end{bmatrix} = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \quad (4.18)$$

$$S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix} \quad (4.19)$$

$$\tilde{A} = \begin{bmatrix} -5 & -6 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad (4.20)$$

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.21)$$

Step 4: From Eq. (2.35) $R(s) = \hat{C} S(s)$. Thus, substituting for \hat{C} and $S(s)$ from Equations (4.17) and (4.19), respectively

$$R(s) = \begin{bmatrix} 1 & 1 & 0 \\ 10 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s+1 & 0 \\ 2(s+5) & 1 \end{bmatrix} \quad (4.22)$$

Now, from Eq. (2.36)

$$P(s) = (\tilde{B}G)^{-1} \{S^{\sigma} - (\tilde{A} + \tilde{B}\hat{F}) S(s)\} \quad (4.23)$$

Let

$$(\tilde{B}G)^{-1} = D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \quad (4.24)$$

and

$$(\tilde{B}\tilde{G})^{-1}(\tilde{A} + \tilde{B}\tilde{F}) = -H = - \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix} \quad (4.25)$$

From Equations (4.23), (4.24), and (4.25), one obtains

$$P(s) = \begin{bmatrix} d_{11}s^2 + h_{12}s + h_{11} & d_{12}s + h_{13} \\ d_{21}s^2 + h_{22}s + h_{21} & d_{22}s + h_{23} \end{bmatrix} \quad (4.26)$$

Thus, the transfer matrix of the multivariable system in its most general form as given by Eq. (2.34) is as follows:

$$T(s) = \begin{bmatrix} s+1 & 0 \\ 2(s+5) & 1 \end{bmatrix} \begin{bmatrix} d_{11}s^2 + h_{12}s + h_{11} & d_{12}s + h_{13} \\ d_{21}s^2 + h_{22}s + h_{21} & d_{22}s + h_{23} \end{bmatrix}^{-1} \quad (4.27)$$

4.5.1 The Problem of Decoupling

In Section 2.6, it was shown that if only the decoupled transfer matrix model is desired, then all the relevant information is obtained from the knowledge of $R(s)$, the system feedback invariant matrix, and the necessary and sufficient condition for decoupling of the system [Falb and Wolowich, 1967; Gilbert, 1969].

Step 5d: If the above mentioned test for decoupling is applied to the system in Eq. (4.11) or the transformed system in Eq. (4.16) and (4.17), the following information is obtained:

- a) The system is decouplable.
- b) The pole zero excess for both subsystems is 1.

Step 6d: Now if each row of the matrix $R(s)$ as obtained in Eq. (4.22) is written as in Eq. (2.54), the following is obtained:

$$\begin{aligned} R_1(s) &= [s+1 \quad 0] = (s+1)[1 \quad 0] \\ R_2(s) &= [2(s+5) \quad 1] = 1[2(s+5) \quad 1] \end{aligned} \quad (4.28)$$

Thus $(s+1)$ and 1 are the highest degree polynomials common to all entries of the first and second rows of $R(s)$ respectively. Since the zeros of the highest degree common polynomial of i th row of $R(s)$ are the zeros of i th subsystem the following information is obtained:

- a) The zeros of the first subsystem are the zeros of $s+1$.
- b) The second subsystem has no zeros.

Once again, since the order of i th subsystem is the sum of the number of zeros and pole zero excess, the following conclusion can be drawn easily.

- a) The order of the first subsystem is $1+1 = 2$.
- b) The order of the second subsystem is $0+1 = 1$.

Step 7d: Since the sum of the orders of the two subsystems $= 2+1 = 3$ is equal to the order of the overall system of Eq. (4.11), there are no uncontrollable poles. This fact could also be verified by checking the

matrix $R'(s)$ of Eq. (2.62). The uncontrollable poles are the zeros of $|R'(s)|$. For this example

$$|R'(s)| = \begin{vmatrix} 1 & 0 \\ 2(s+5) & 1 \end{vmatrix} = 1$$

Since the polynomial $|R'(s)|$ has no zeros, there are no uncontrollable poles as stated before.

Step 8d: In summary, if decoupling is desired then choose the model transfer matrix such that the first subsystem has a zero at $s = -1$ and two arbitrary poles and the second subsystem has no zero but one arbitrary pole.

Step 9d: Hence, given a transfer matrix model which meets the above specifications, the pair $\{F, G\}$, to realize the model response from the plant, could easily be found by using any one of the following methods:

1. Original algorithm of Gilbert [1969] for decoupling
2. Wolowich's [1972] algorithm for exact model matching
3. Design method described in this study.

All of the above algorithms give the same F and G . Method 2 is computationally more difficult because, as mentioned in Sec. 2.5, it requires that the model transfer matrix $T(s)$ be put into the form

$$T_m(s) = R_m(s) P_m^{-1}(s),$$

where $R_m(s)$ and $P_m(s)$ are relatively right prime polynomial matrices as given in Eq. (2.38).

The algorithm for finding the feedback and gain matrices F and G is illustrated for the general case of model matching which is considered next.

4.5.2 The General Case of Model Matching

The general case of model matching is illustrated by trying to approximately match the plant transfer matrix to a given model transfer matrix using state variable feedback. Assume that the following model meets the design requirements for the given plant of Eq. (4.11).

$$T(s) = \begin{bmatrix} \frac{6(s+1)}{(s+2)(s+3)} & \frac{0.6(s+1)}{(s+2)(s+3)} \\ \frac{10}{(s+2)(s+5)} & \frac{4}{(s+4)} \end{bmatrix} \quad (4.29)$$

If one were to test for exact model matching, it would be found that the necessary and sufficient conditions for exact model matching as given in Theorem 2.2 [Wolowich, 1972] are not satisfied, and, hence, the plant of Eq. (4.11) cannot be matched to the above model. Also, application of the test does not give any hint as to what changes should be made in the model to force realizability. Thus, the designer is left to his luck and experience to try one model after another until he finds one that can be matched and that meets his requirements.

The design method described in the previous section is now applied to investigate if there is any other model which matches the plant of Eq. (4.11) and at the same time, approximates the model of Eq. (4.29).

Step 7: Substituting for K_1 and R_1 from Eq. (4.30) and (4.31), one obtains by using Eq. (3.20)

$$P_1 = K_1^{-1} R_1$$

Hence,

$$\begin{aligned} P_1 &= \begin{bmatrix} 1.0 & 0.1 \\ 1.0 & 1.0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -.111 \\ 10.0 & 1.111 \end{bmatrix} \end{aligned} \quad (4.32)$$

Step 8: To check for the component definiteness of the polynomial matrix $P(s)$, expand $P(s)$ as obtained in Eq. (4.26) in its components as described in Eq. (3.8). Clearly,

$$P(s) = P_1 + P_2 s + P_3 s^2$$

Since the σ_i 's are greater than or equal to 1, P_1 and P_2 do not have any zero columns, and hence, one need not form Γ_1 and Γ_2 . Since the last non-zero component of $P(s)$ is P_3 , one needs to form and check Γ_3 only. From Eq. (4.26)

$$P_3 = \begin{bmatrix} d_{12} & 0 \\ d_{22} & 0 \end{bmatrix} \quad (4.33)$$

and, hence, only the second column of P_3 is zero. Thus, according to definition 4.1, Γ_3 is a 1×1 submatrix formed by the elements at the cross section of the second row and the second column of P_1 . Thus,

$$\Gamma_3 = (p_{22})_1 = 1.111$$

and, hence, Γ_3 is nonsingular and thus $P(s)$ is component definite matrix.

Step 9: Now find P_2 using Eq. (3.20)

$$K_1 P_2 = R_2 + K_2 P_1 \quad (3.20)$$

or

$$P_2 = K_1^{-1} \{R_2 + K_2 P_1\}$$

After substituting for K_1 , P_1 , K_2 and R_2 , the result is

$$P_2 = \begin{bmatrix} 1.0 & 0.1 \\ 1.0 & 1.0 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -0.167 & -0.0167 \\ 0.70 & 0.250 \end{bmatrix} \begin{bmatrix} 0 & -0.111 \\ 10.0 & 1.111 \end{bmatrix} \right\}$$

Hence, from above

$$\begin{aligned} P_2 &= \begin{bmatrix} 1.0 & 0.1 \\ 1.0 & 1.0 \end{bmatrix}^{-1} \begin{bmatrix} 0.833 & 0 \\ 2.50 & 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.426 & -0.022 \\ 4.07 & 0.222 \end{bmatrix} \end{aligned} \quad (4.34)$$

Now, from Eq. (4.6), $P(s)$ is column proper if

$$\begin{bmatrix} p_{\sigma_1+1}^1 & p_{\sigma_2+1}^2 \end{bmatrix}$$

contains all independent vectors, i.e., the columns of the following matrix

$$\begin{bmatrix} 1 & p_2^2 \\ p_3 & p_2 \end{bmatrix} \quad (4.35)$$

must be independent vectors. Since p_2^2 is known, one must check that it be a non-zero vector. From Eq. (4.34) it is seen to be a non-zero vector; hence, proceed to find P_3 .

Solving Eq. (4.8) for $r=2$, one obtains

$$K_1 P_3 = Q_2 + K_3 P_1 \quad (4.36)$$

and

$$Q_2 = R_3 + K_2 P_2$$

Since R_3 , K_2 and P_2 are all known at this point, Q_2 is obtained by direct substitution as follows:

$$Q_2 = [0] + \begin{bmatrix} -.1667 & -.0167 \\ 0.70 & 0.250 \end{bmatrix} \begin{bmatrix} 0.426 & -.022 \\ 4.07 & 0.222 \end{bmatrix}$$

i.e.,

$$Q_2 = \begin{bmatrix} -.139 & 0 \\ 1.32 & 0.04 \end{bmatrix}$$

Next, from Eq. (4.30), since the second column of K_3 is unspecified, this must be determined by considering the second column of both sides of Eq. (4.36).

$$K_1 p_3^2 = q_2^2 + K_3 p_1^2$$

But, from Eq. (4.31), $p_3^2 = 0$ and hence

$$K_3 p_1^2 = -q_2^2$$

or

$$\begin{bmatrix} 1 & 2 \\ k_3 & k_3 \end{bmatrix} \begin{bmatrix} (p_{12})_1 \\ (p_{22})_1 \end{bmatrix} = -q_2^2$$

The above can be partitioned into known and unknown parts as shown in Eq. (4.10).

$$k_3^1 (p_{12})_1 + k_3^2 (p_{22})_1 = -q_2^2$$

Hence

$$k_3^2 (p_{22})_1 = -q_2^2 - k_3^1 (p_{12})_1$$

and thus

$$k_3^2 = - \left\{ q_2^2 + k_3^1 (p_{12})_1 \right\} \left[(p_{22})_1 \right]^{-1}$$

Substituting for all quantities on the right hand side of the above expression, one obtains

$$\begin{aligned}
 k_3^2 &= - \left\{ \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.306 \\ -.390 \end{bmatrix} [-0.111] \right\} [1.111]^{-1} \\
 &= - \begin{bmatrix} -.0340 \\ 0.0833 \end{bmatrix} (1.11)^{-1} \\
 &= \begin{bmatrix} 0.0306 \\ -.0750 \end{bmatrix}
 \end{aligned}$$

The remaining first column of P_3 (the only unknown part) is now calculated easily from Eq. (4.36) by either solving for the first column of P_3 or by solving for the complete P_3 , even though the latter involves unnecessary more computation. So consider the first column of both sides of Eq. (4.36):

$$\begin{aligned}
 K_1 P_3^1 &= q_2^1 + K_3 P_1^1 \\
 &= \begin{bmatrix} -0.139 \\ 1.32 \end{bmatrix} + \begin{bmatrix} 0.306 & 0.0306 \\ -.39 & -.0750 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} \\
 &= \begin{bmatrix} -0.139 \\ 1.32 \end{bmatrix} + \begin{bmatrix} 0.306 \\ -.750 \end{bmatrix} = \begin{bmatrix} 0.167 \\ 0.567 \end{bmatrix}
 \end{aligned}$$

C-2

Hence

$$\begin{aligned}
 p_3^1 &= K_1^{-1} \begin{bmatrix} 0.167 \\ 0.667 \end{bmatrix} \\
 &= \begin{bmatrix} 1.0 & 0.1 \\ 1.0 & 1.0 \end{bmatrix}^{-1} \begin{bmatrix} 0.167 \\ 0.567 \end{bmatrix} \\
 &= \begin{bmatrix} 0.123 \\ 0.444 \end{bmatrix}
 \end{aligned}$$

Now, from Eq. (4.35), p_3^1 must be independent of p_2^2 . Since p_2^2 is the second column of P_2 in Eq. (4.34), by inspection p_2^2 and p_3^1 are mutually independent column vectors.

Except for the determination of the necessary F and G , the design is thus complete, since $P(s)$ is completely known and is guaranteed to satisfy the generalized error coefficients K_1 , K_2 , and K_3 as specified in Eq. (4.30). Next, check the designed transfer matrix so obtained.

Step 10: The polynomial matrix $P(s)$ is completely specified as follows:

$$\begin{aligned}
 P(s) &= P_1 + P_2 s + P_3 s^2 \\
 &= \begin{bmatrix} .123s^2 + .426s & -(0.0222s + .111) \\ .444s^2 + 4.07s + 10 & .222s + 1.111 \end{bmatrix} \quad (4.37)
 \end{aligned}$$

Hence

$$P^{-1}(s) = P^*(s)/\Delta$$

where

$$P^*(s) = \text{adj } P(s) = \begin{bmatrix} .222(s+5) & .0222(s+5) \\ -(.444s^2 + 4.07s + 10) & .123s^2 + .426s \end{bmatrix} \quad (4.38)$$

$$\begin{aligned} \Delta = |P(s)| &= (.123s^2 + .426s)(.222)(s+5) + (.444s^2 + 4.07s + 10)(.022)(s+5) \\ &= .022 (s+5) \{1.23s + 4.26s + .444s + 4.07s + 10\} \\ &= .022 (s+5) \{1.674 s^2 + 8.33s + 10\} \\ &= .0371 (s+5) \{s^2 + 5s + 6\} \\ &= .0371 (s+5)(s+2)(s+3) \end{aligned}$$

From Eq. (2.34)

$$\begin{aligned} T(s) &= R(s) P^{-1}(s) \\ &= R(s) P^*(s) / \Delta \end{aligned}$$

Substituting for $R(s)$ and $P^*(s)$ from Equations (4.22) and (4.38), respectively

$$\begin{aligned} R(s)P^*(s) &= \begin{bmatrix} (s+1) & 0 \\ 2(s+5) & 1 \end{bmatrix} \begin{bmatrix} -.222(s+5) & .0222(s+5) \\ -(.444s^2 + 4.07s + 10) & .123s^2 + .426s \end{bmatrix} \\ &= \begin{bmatrix} .222(s+1)(s+5) & .0222(s+1)(s+5) \\ .444(s+5)^2 - .444s^2 - 4.07s - 10 & .0444(s+5)^2 + .123s^2 + .426s \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} .222 (s+1)(s+5) & .0222 (s+1)(s+5) \\ .37s + 1.111 & .167s^2 + .870s + 1.111 \end{bmatrix}$$

Therefore

$$T(s) = R(s) P^*(s)/\Delta$$

$$= \begin{bmatrix} \frac{.222 (s+1)(s+5)}{.0371 (s+5)(s+2)(s+3)} & \frac{.0222 (s+1)(s+5)}{.0371 (s+5)(s+2)(s+3)} \\ \frac{.37(s+3)}{.0371 (s+5)(s+2)(s+3)} & \frac{.167 (s+2.22)(s+2.98)}{.0371 (s+5)(s+2)(s+3)} \end{bmatrix}$$

$$\approx \begin{bmatrix} \frac{6(s+1)}{(s+2)(s+3)} & \frac{0.6 (s+1)}{(s+2)(s+3)} \\ \frac{10}{(s+2)(s+5)} & \frac{4.5(s + 2.22)}{(s+5)(s+2)} \end{bmatrix} \quad (4.39)$$

Some comments are in order before the pair $\{F, G\}$ are found to realize the transfer matrix of Eq. (4.39). The transfer matrix actually desired is given in Eq. (4.29). A comparison between the transfer matrices of Equations (4.29) and (4.39) shows that except for the transfer function entry $t_{22}(s)$ all other entries are exactly realized. Also, for the transfer function $t_{22}(s)$, the difference between desired and obtained is not intolerable, as can be seen by comparing the time response of the two as shown in Fig. (4.2).

However, one may not always be as lucky as in this example. As mentioned in the beginning, this is a relatively simple example to demonstrate the design method of multivariable systems. It is quite possible and probable that the response obtained would not be as close

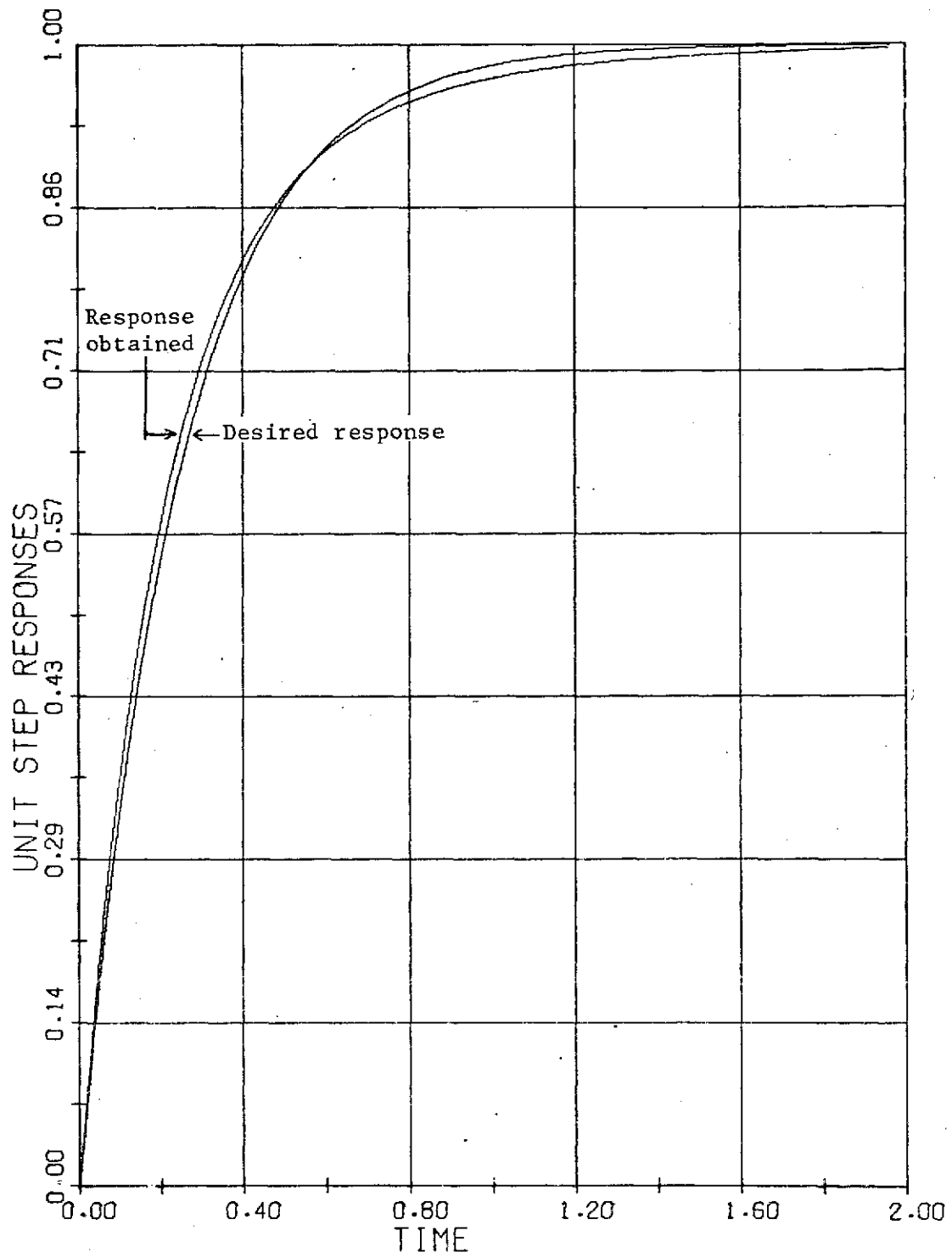


Fig. 4.2 Comparison of Time Responses in Example 4.1

to the desired response as indicated in Fig. 4.2 for this case. For this example one could guess to some extent which of the transfer function entry would differ most in its response. Since the second column of K_3 is not specifiable, one would guess some change in the second column of $T(s)$. Also, since $t_{12}(s)$ is just a constant multiplication of $t_{11}(s)$, one would not expect much change in $t_{12}(s)$. Thus, the only entry left is $t_{22}(s)$ which might have much different transient response than desired. As can be seen by examining $t_{21}(s)$ and $t_{22}(s)$ in Eq. (4.39), they both have closed loop poles at the same position. (Thus, it seems $(k_{21})_3$ has more effect on $t_{22}(s)$ than any other single error coefficients.) The above comments are included to give the reader some insight to the problem solution. However, extreme caution should be taken before any of the above concepts are generalized.

Once again, it should be noted that even if the desired transfer function and the one obtained might differ significantly in their transient behavior, they still asymptotically coincide in their steady state behavior if the system design is stable.

Step 11: To find the pair $\{\hat{F}, G\}$ which realizes the transfer matrix as obtained in Eq. (4.39), find the pair $\{F, G\}$ to realize $P(s)$ as obtained in Eq. (4.37).

Comparing Eq. (4.26) and (4.37), the D matrix of Eq. (4.24) is given as follows:

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} .123 & -.0222 \\ .444 & .222 \end{bmatrix}$$

Hence, from Eq. (4.24)

$$(\tilde{B}G) = D^{-1} = \begin{bmatrix} .123 & -.022 \\ .444 & .222 \end{bmatrix}^{-1} = \begin{bmatrix} 6.0 & .60 \\ -12.0 & 3.3 \end{bmatrix} \quad (4.40)$$

Hence, $G = \tilde{B}^{-1}D^{-1} = D^{-1}$, because from Eq. (4.21) $\tilde{B} = \text{Identity matrix}$.

Thus

$$G = \begin{bmatrix} 6.0 & .60 \\ -12.0 & 3.3 \end{bmatrix} \quad (4.41)$$

Once again comparing Equations (4.26) and (4.37), the H matrix of Eq. (4.25) is given by

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix} = \begin{bmatrix} 0 & .426 & -.11 \\ 10 & 4.07 & 1.11 \end{bmatrix} \quad (4.42)$$

But, from Eq. (4.25)

$$(\tilde{B}G)^{-1} (\tilde{A} + \tilde{B}\hat{F}) = -H$$

Hence

$$\tilde{A} + \tilde{B}\hat{F} = -(\tilde{B}G)H$$

This implies

$$\begin{aligned} \hat{F} &= -\tilde{B}^{-1} \{ (\tilde{B}G)H + \tilde{A} \} \\ &= -GH - \tilde{B}^{-1} \tilde{A} \end{aligned}$$

After substituting for G, H, \tilde{B} and \tilde{A} from Equations (4.41), (4.42), (4.21) and (4.20), respectively, one gets

$$\begin{aligned}
 \hat{F} &= - \begin{bmatrix} 6.0 & .6 \\ -12.0 & 3.3 \end{bmatrix} \begin{bmatrix} 0 & .426 & -.11 \\ 10 & 4.07 & 1.11 \end{bmatrix} \\
 &= - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
 &= - \begin{bmatrix} 6.0 & 5.0 & 0 \\ 33.0 & 8.37 & 5 \end{bmatrix} - \begin{bmatrix} -5 & -6 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} -1.0 & 1.0 & 0 \\ -33.0 & -8.37 & -3.0 \end{bmatrix} \tag{4.43}
 \end{aligned}$$

Step 12: Using Eq. (2.14)

$$F = \hat{F}Q \tag{2.14}$$

After substituting for \hat{F} and Q from Equations (4.43) and (4.15) respectively, one obtains

$$\begin{aligned}
 F &= \begin{bmatrix} -1.0 & 1.0 & 0 \\ -33.0 & -8.37 & -30 \end{bmatrix} \begin{bmatrix} -.25 & .125 & 0 \\ 1.25 & -.125 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1.50 & -.25 & 0 \\ -2.2 & -3.08 & -3.0 \end{bmatrix} \tag{4.44}
 \end{aligned}$$

Thus, Equations (4.44) and (4.41) give the required feedback pair $\{F, G\}$ which realizes the transfer matrix of Eq. (4.39).

This satisfactorily completes the design. The state variable feedback compensated plant is shown in Fig. (4.3). In the next section the results of this chapter are summarized, and advantages and disadvantages are pointed out.

4.6 Conclusion

A completely new design method for the design of multivariable systems is presented. The main advantage of the above design method is that it realizes a given (pre-specified) model transfer matrix approximate, where necessary and sufficient conditions for exact model matching fail to produce any solution. The design method is based upon approximating each element of the transfer matrix by a finite element series. The coefficients of the elements in the series are well known to have direct relationship with the generalized error coefficients. As compared to exact model matching, this design method tries to realize generalized error coefficients of the model. It is shown that certain sufficiency conditions (Theorem 4.1) must be met which in turn restrict the allowable

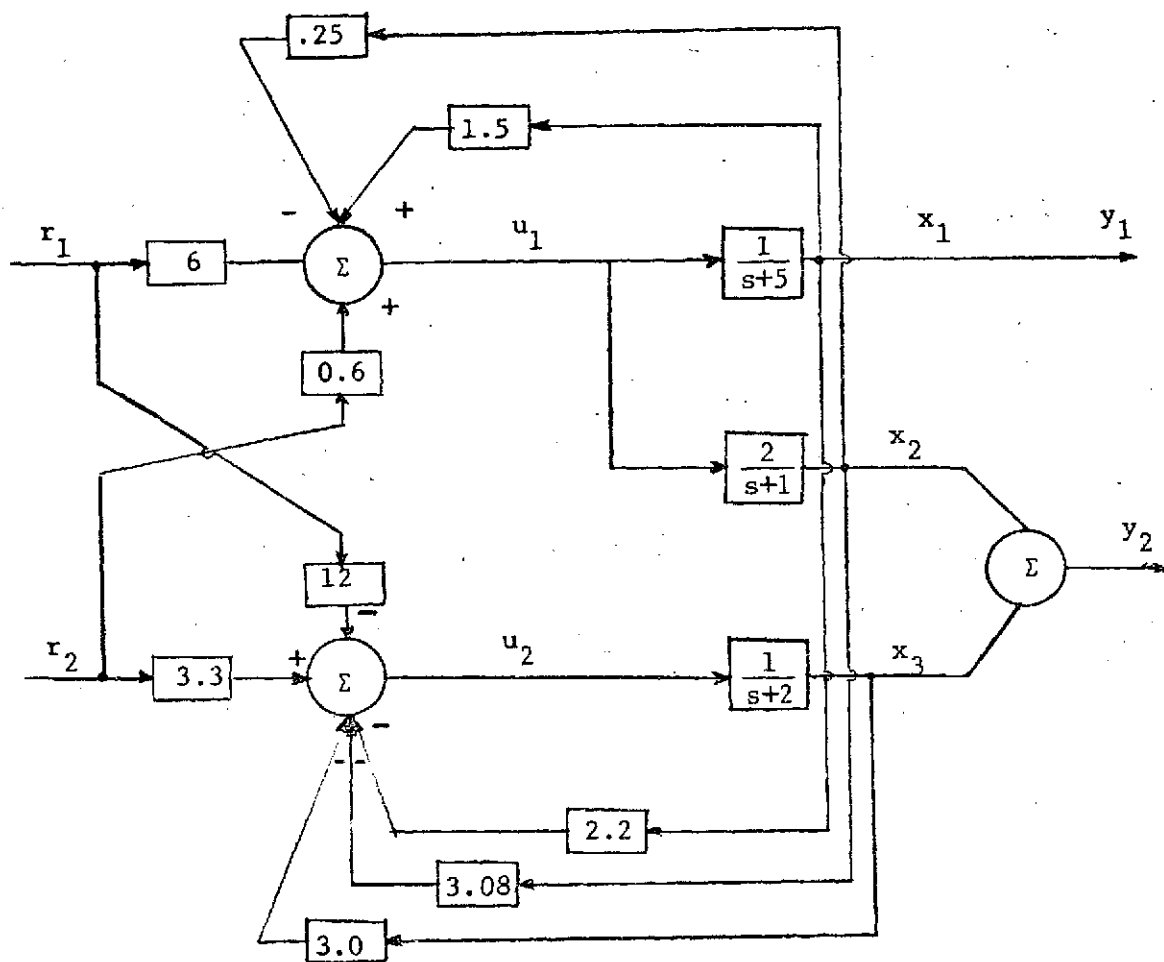


Fig. 4.3 Compensated System

range of error coefficients. This in itself is not a serious design restriction for all practical purposes.

The main disadvantage of the design method is that the design does not take into account the transient behavior of the system. This problem could theoretically be serious enough to cause instability. This disadvantage is attributed to the fact that an n terms series representation of an n th order transfer function element does not uniquely determine the transfer function. That is to say, there are many transfer functions, some of them having undesirable transient behavior or, in the worst case, representing unstable systems, which have the same n generalized error coefficients.

Thus, it is possible that the application of the above design procedure may result in an undesirable transient response. Of course this fact may be checked easily by comparing the desired and actual step function responses. In the next chapter a second design method is given which ensures stability on the risk of deteriorating the steady state behavior of some of the elements in transfer matrix. Thus, even though an overall satisfactory solution is not guaranteed in advance, the design method of this chapter gives valuable insight into the solution to the problem of model matching.

CHAPTER 5

THE POLE FIXING METHOD

5.1 Introduction and Organization of the Chapter

The pole fixing method for the design of multivariable systems is the subject matter of this chapter. The pole fixing method has a distinct advantage over the error coefficient design method of the last chapter. The shortcomings of the error coefficient method are summarized and the salient features of the pole fixing methods are discussed in the next section. The constraints of the pole fixing method are described via Theorem 5.1 in Section 5.3. The proof of the theorem is constructive and leads to a design procedure. The step-by-step design procedure is described in Section 5.4. In Section 5.5 the pole fixing design procedure is illustrated by reworking the example of the last chapter. Some special cases of the pole fixing method are discussed in the next section. Finally, the findings of the chapter are summarized in Section 5.7.

5.2 Need for Pole Fixing

The error coefficient design method of the last chapter is based completely upon the generalized error coefficient representation of a transfer function. The generalized error coefficients describe the system input behavior only after sufficient time has elapsed for the system transients to decay to insignificant amplitudes. Thus, even though the

error coefficient design method of the last chapter is simple, it does not ensure satisfactory transient behavior of the systems. In the worst case the design may even lead to an unstable system.

The pole fixing method to be introduced in this chapter ensures the system stability by prefixing all the system poles at specified locations in the left hand S plane. It is well known [Anderson and Luenberger, 1967; Wonham, 1967; Davison, 1968; and Sridhar and Lindorff, 1972] that if the multivariable system of Eq. (2.1) is completely controllable, then all of the n poles of the system can be fixed arbitrarily by using l.s.v.f. alone. The pole fixing method, in addition to assigning all the poles of the system, fixes zeros and steady state errors for some of the transfer functions in the transfer matrix. However, the fixing of the zeros is done indirectly via error coefficients, and requires that R_1 , the first component of the system invariant matrix $R(s)$, satisfy certain sufficiency conditions. Most of the time these conditions can be met by simple changes in the names of inputs and outputs.

Compared to the decoupling technique, the pole fixing method has three distinct advantages:

1. considerably more freedom for the off-diagonal terms is achieved,
2. controllable systems which cannot be decoupled can be designed for undecoupled response, and
3. there are no uncontrollable poles.

Compared to the error coefficient method of the last chapter, it has the advantage that the transient behavior can be controlled as well. The

only disadvantage is that no error coefficients can be specified for one side of the diagonal terms in the transfer function matrix which amounts to not being able to control the zeros of that side of the transfer function matrix, directly or indirectly. However, if the open loop system has only one way coupling, i.e., $R(s)$ is a triangular matrix, or can be made a triangular matrix by changing the names of inputs and outputs, then one side of the diagonal terms in the transfer matrix could be made identically zero. This is discussed in Section 5.6 under the special case of triangular decoupling.

Thus, despite the restrictions that the zeros for only one side of the diagonal terms in the transfer function matrix can be specified, the pole fixing design method is useful for two reasons. First, it gives a lead into how the transient behavior specifications can be accommodated in the design criteria, and, secondly, information is obtained as to what is realizable from the system.

In the next section the constraints of the pole fixing method are mentioned in Theorem 5.1. The proof of the theorem leads to a design procedure.

5.3 The Pole Fixing Method

In this section the design constraints are introduced through Theorem 5.1. The design procedure follows from the proof of the theorem. First, a definition is introduced to help understand the implications of the theorem that follows:

Definition 5.1: Lower (upper) Definite Matrix: An $m \times m$ constant matrix P is called Lower (upper) definite if the $\gamma \times \gamma$ submatrix formed from the lower right most (upper left most) terms of P is nonsingular for all $1 \leq \gamma \leq m$.

A matrix which is both lower definite and upper definite is simply called definite nonsingular matrix. Conversely, if a matrix is definite nonsingular, it is both lower definite and upper definite. Notice that any triangular matrix (one side of diagonal terms identically zero) if nonsingular is a definite nonsingular matrix, a requirement which is satisfied if, and only if, none of the diagonal terms are zero.

Theorem 5.1: If the multivariable system is controllable, then all the n poles of the overall system can be placed arbitrarily via l.s.v.f. along by forcing $P(s)$ in Eq. (2.36) to be triangular. Moreover, if $R_1 = R(s)|_{s=0}$ is lower (upper) definite, where $R(s)$ is the system invariant matrix of Eq. (2.35), then the following freedom in the choice of error coefficient is maintained by choosing $P(s)$ lower (upper) triangular.

- (1) Elements of K_1 can be chosen arbitrarily, for $i \geq j$ ($i \leq j$), provided the elements $(k_{ij})_1$ of K_1 are so chosen as to force K_1 to be a lower (upper) definite matrix.
- (2) σ_j addition error coefficients can be arbitrarily assigned for those elements $t_{ij}(s)$ of the overall transfer matrix $T(s)$ for which $i > j$ ($i < j$).

Throughout the statement of the above theorem, the expression in parenthesis corresponds to R_1 being an upper definite matrix. Notice that if R_1 is definite nonsingular, (i.e., is both lower and upper nonsingular) then $P(s)$ could be chosen either upper triangular or lower triangular. This results in additional design freedom, as illustrated in the example of Section 5.5.

Also notice that since $P(s)$ is forced to be triangular and since the i th diagonal element in $P(s)$ is chosen to be a polynomial of degree σ_i (essential for choosing all n poles), $P(s)$ is automatically forced to be column proper. Thus, $P(s)$ as specified above is realizable by linear state variable feedback alone.

Proof of Theorem 5.1: The first part of the theorem is proved by noticing that if the system is controllable, then the Lunenberger transformation of Section 2.3 can be found and the overall transfer matrix of the system can be written as $T(s) = R(s) P^{-1}(s)$ as given by Eq. (2.34). Thus, all n poles can be placed arbitrarily simply by forcing $P(s)$ triangular with i th diagonal term a polynomial of degree σ_i , where the σ_i are specified by Eq. (2.8). But such a $P(s)$ is realizable by l.s.v.f. alone because it is column proper and i th column has the i th diagonal term as the highest degree polynomial in it. Thus, the l.s.v.f. pair (F,G) can be found by simply equating the desired $P(s)$ with the one in Eq. (2.36).

The second part of the theorem is proved by showing that the off diagonal nonzero terms in $P(s)$ can be used to specify

the error coefficients of the corresponding terms in the transfer matrix. Conversely, it is shown that the off diagonal nonzero terms in $P(s)$ are calculated for any arbitrary choice of the error coefficients as specified in the theorem. This is accomplished in two parts. First, the matrix P_1 , the first component of $P(s)$, is determined. Notice that to specify all n poles, the diagonal terms in the triangular matrix $P(s)$ are specified to within a constant only. This gives extra freedom, i.e., the first error coefficient can be specified even for the diagonal terms in $T(s)$. Thus, P_1 is determined first. Next, the remaining components of $P(s)$ are determined one by one. Notice that only terms on one side of the diagonal need be determined because the diagonal terms are now completely specified and the terms on the other side of diagonal terms are specified to be identically zero.

The theorem is proved for R_1 an upper definite matrix. The proof for R_1 lower definite can be developed on the same lines. For R_1 upper definite, $P(s)$ is forced to be upper triangular and, hence, P_1, P_2, \dots are all upper triangular matrices.

To determine P_1 , consider Eq. (3.19) for $r = 0$, which is as follows:

$$K_1 P_1 = R_1 \quad (3.19)$$

where K_1, P_1 and R_1 are all $m \times m$ constant matrices.

Now, consider the i^{th} column of both sides of the above equation. Since P_1 is an upper triangular matrix, the i^{th} column is simply

$$[k^1 \dots k^m]_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [r^i]_1$$

The above could be simplified as follows, where zero terms have been dropped:

$$[k^1 \dots k^i]_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \end{bmatrix} = [r^i]_1$$

Next, consider the top i rows and the remaining $m-i$ rows of the above equation separately as follows:

$$\begin{bmatrix} k_{11} & \dots & k_{1i} \\ \vdots & & \vdots \\ k_{i1} & \dots & k_{ii} \end{bmatrix}_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \end{bmatrix}_1 = \begin{bmatrix} r_{1i} \\ \vdots \\ r_{ii} \end{bmatrix}_1 \quad (5.1)$$

and

$$\begin{bmatrix} k_{i+1 \ 1} & \dots & k_{i+1 \ i} \\ \vdots & & \vdots \\ k_{m \ 1} & \dots & k_{m \ i} \end{bmatrix}_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \end{bmatrix}_1 = \begin{bmatrix} r_{i+1i} \\ \vdots \\ r_{mi} \end{bmatrix}_1 \quad (5.2)$$

Now, Equations (5.1) and (5.2) must be solved for $i = 1, 2 \dots m$.

Substitution of $i = 1$ in Equations (5.1) and (5.2), respectively gives

$$(k_{11})_1 (p_{11})_1 = (r_{11})_1$$

and

$$\begin{bmatrix} k_{21} \\ \vdots \\ k_{m1} \end{bmatrix}_1 (p_{11})_1 = \begin{bmatrix} r_{21} \\ \vdots \\ r_{m1} \end{bmatrix}_1$$

Since $R(s)$ and, hence, R_1 are already known, $(p_{11})_1$ and $\begin{bmatrix} k_{12} \\ \vdots \\ k_{1m} \end{bmatrix}_1$ are

are uniquely determined for any nonzero choice of $(k_{11})_1$.

Equation (5.1) can next be solved for $i = 2$ to give

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_1 \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}_1 = \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix}_1$$

from which $\begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}_1$ is uniquely determined for any choice of $(k_{12})_1$ and

$(k_{22})_1$ which keeps $\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_1$ a nonsingular matrix. Notice that since

R_1 is upper definite matrix $\begin{bmatrix} r_{11} \\ r_{12} \end{bmatrix}_1$ is independent of $\begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix}_1$. Hence, by

considering the first two rows and top two columns of Eq. (3.19), it is

easily seen that $\begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}_1$ is independent of $\begin{bmatrix} p_{11} \\ 0 \end{bmatrix}_1$ which in turn implies $(p_{22})_1 \neq 0$.

If $m > 2$, one would need to solve Eq. (5.2) for the remaining unspecifiable elements in the second column of k_1 . This can be done by splitting the left side of Eq. (5.2) into two parts as follows

$$\begin{bmatrix} k_{i+1,1} & \cdots & k_{i+1,i-1} \\ \vdots & & \\ k_{m,1} & \cdots & k_{m,i-1} \end{bmatrix}_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1,i} \end{bmatrix}_1 + \begin{bmatrix} k_{i+1,i} \\ \vdots \\ k_{m,i} \end{bmatrix}_1 (p_{ii})_1 = \begin{bmatrix} r_{i+1,i} \\ \vdots \\ r_{m,i} \end{bmatrix}_1$$

and the above can be rewritten as follows

$$\begin{bmatrix} k_{i+1,i} \\ \vdots \\ k_{m,i} \end{bmatrix}_1 = \frac{1}{(p_{ii})_1} \left\{ \begin{bmatrix} r_{i+1,i} \\ \vdots \\ r_{m,i} \end{bmatrix}_1 - \begin{bmatrix} k_{i+1,1} & \cdots & k_{i+1,i-1} \\ \vdots & & \\ k_{m,1} & \cdots & k_{m,i-1} \end{bmatrix}_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1,i} \end{bmatrix}_1 \right\} \quad (5.3)$$

Also, Eq. (5.1) can be rewritten as follows

$$\begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \end{bmatrix}_1 = \begin{bmatrix} k_{11} & \dots & k_{1i} \\ \vdots & & \vdots \\ k_{i1} & & k_{ii} \end{bmatrix}^{-1} \begin{bmatrix} r_{1i} \\ \vdots \\ r_{ii} \end{bmatrix}_1 \quad (5.4)$$

Thus solving Eq. (5.3) for $i = 2$, gives the remaining elements of the second column of k_1 .

The above result can be used to solve Eq. (5.4) for $i = 3$, and the results could then be used to calculate the remaining elements of the third column of k_1 , if $m > 3$. This sequence is repeated until Eq. (5.4) is solved for $i = m$ which gives the desired upper triangular p_1 for the desired upper triangular portion of k_1 (including the diagonal terms).

Once again, it can be seen from Eq. (5.4) that a unique solution of Eq. (5.4) requires that elements in K_1 be chosen such that K_1 is an upper definite matrix. Similarly, a unique solution of Eq. (5.3) requires that $(p_{ii})_1 \neq 0$, which is obtained only if R_1 is upper definite. This proves part (1) of Theorem (5.1).

To prove the second part of the theorem, that is, that the remaining components of $P(s)$ are uniquely determined by an arbitrary choice of corresponding components of $T(s)$, consider Eq. (3.19) for $r \geq 1$, which is given below as

$$K_1 P_{r+1} = Q_r + K_{r+1} P_1 \quad (3.19)$$

Notice that if the above equation is solved in sequence for $r = 1, 2, \dots$, then Q_r is always known. Also, P_1 and K_1 are completely known from above, and the diagonal terms in P_{r+1} are completely known because of the fact that the diagonal terms in $P(s)$ are now completely specified. Finally, P_{r+1} is upper triangular because $P(s)$ is assumed to be upper triangular. Thus, it suffices to prove that the terms above the diagonal term in P_{r+1} and terms below and including diagonal terms in K_{r+1} are uniquely determined for arbitrarily specified terms above the diagonal in K_{r+1} . To see this, let $j = r+1$ and rewrite the above equation as follows:

$$K_1 P_j = Q_r + K_j P_1 \quad (5.5)$$

Consider the i th column of both sides of the above equation which is given as follows:

$$\begin{bmatrix} k_1^1 & \dots & k_1^m \\ \vdots & & \vdots \\ p_{1i} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_j = [q_r^i] + \begin{bmatrix} k_j^1 & \dots & k_j^m \\ \vdots & & \vdots \\ p_{ji} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_1 \quad (5.6)$$

Using identity (4.1), the above equation could be written as follows:

$$[k_1^1 \dots k_1^{i-1}] \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1i} \end{bmatrix}_j + [k_1^i] (p_{ii})_j + [k_1^{i+1} \dots k_1^m] \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_j = [q_r^i] +$$

$$[k_j^1 \dots k_j^{i-1}] \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1i} \end{bmatrix}_1 + [k_j^i] (p_{ii})_1 + [k_j^{i+1} \dots k_j^m] \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_1$$

(5.7)

which can be simplified to give

$$[k_1^1 \dots k_1^{i-1}] \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1i} \end{bmatrix}_j + [k_1^i] (p_{ii})_j = [q_r^i] + [k_j^1 \dots k_j^{i-1}] \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1i} \end{bmatrix}_1 + [k_j^i] (p_{ii})_1$$

(5.8)

Now, break the above Eq. (5.8) into two parts.

1. Recombine the last two terms of the right side of the equation and consider the top $(i-1)$ rows of the both sides of the equation thus obtained. The following results

$$\begin{aligned}
& \begin{bmatrix} k_{11} & \dots & k_{1 \ i-1} \\ \vdots & & \\ k_{i-1 \ 1} & \dots & k_{i-1 \ i-1} \end{bmatrix}_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1 \ i} \end{bmatrix}_j + \begin{bmatrix} k_{1i} \\ \vdots \\ k_{i-1} \end{bmatrix}_1 (p_{ii})_j = \begin{bmatrix} q_{1i} \\ \vdots \\ q_{i-1 \ i} \end{bmatrix}_r \\
& + \begin{bmatrix} k_{11} & \dots & k_{1i} \\ \vdots & & \vdots \\ k_{i-1 \ 1} & \dots & k_{i-1 \ i} \end{bmatrix}_j \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \end{bmatrix}_1 \quad (5.9)
\end{aligned}$$

2. Again, recombine the first two terms of the left side of Eq. (5.8) and consider the last $m-i+1$ rows of the result. The following equation is obtained.

$$\begin{aligned}
& \begin{bmatrix} k_{i1} & \dots & k_{ii} \\ \vdots & & \\ k_{m1} & \dots & k_{mi} \end{bmatrix}_1 \begin{bmatrix} p_{1i} \\ \vdots \\ p_{ii} \end{bmatrix}_j = \begin{bmatrix} q_{ii} \\ \vdots \\ q_{mi} \end{bmatrix}_r + \begin{bmatrix} k_{i1} & \dots & k_{i \ i-1} \\ \vdots & & \\ k_{m1} & \dots & k_{m \ i-1} \end{bmatrix} \begin{bmatrix} p_{1i} \\ \vdots \\ p_{i-1 \ i} \end{bmatrix} + \begin{bmatrix} k_{ii} \\ \vdots \\ k_{mi} \end{bmatrix}_j (p_{ii})_1 \\
& \quad (5.10)
\end{aligned}$$

Equations (5.8), (5.9), and (5.10), when solved in sequence for $i = 1, 2, \dots, m$, yield a complete solution as follows.

If $[p_j^i]$ the i th column of P_j , is completely specified, then $[k_j^i]$ is completely and uniquely determined by using Eq. (5.8), since everything else in the equation is known and $(p_{ii})_1 \neq 0$.

If $[p_j^i]$ is not known completely, then the only unknown part is the terms above the diagonal term in that column which could be uniquely determined by using Eq. (5.9) for corresponding arbitrarily specified terms in $[k_j^i]$. This is so because all other terms are known in Eq. (5.9), and since K_1 being an upper definite matrix, inverse of its upper sub-matrix exists.

Finally, the remaining portion of the i th column of K_j is uniquely determined from Eq. (5.10) because $(p_{ii})_1 \neq 0$ and all other quantities are known.

Thus, the i th column of P_j and K_j are uniquely determined for arbitrary above the diagonal terms in $[k_j^i]$. Incrementing i and repeating the procedure until $i = m$, completely specifies P_j with K_j as stated in the theorem. Next, increment r , and solve for the unknown part of the next higher component of $P(s)$ until all the components are determined. The $P(s)$ so obtained remains column proper with the degree of the highest degree polynomial in i th column being σ_i . Hence, the l.s.v.f. pair $\{F, G\}$ can be found to realize the above $P(s)$, which in turn forces all the n poles of the system as desired and achieves certain other desired properties as specified in terms of error coefficients.

The theorem is thus proved for the case of R_1 a upper definite matrix. For the case of R_1 a lower definite matrix one can proceed similarly by forcing $P(s)$ to be a lower triangular matrix. The equations involved in solving for the components of $P(s)$ are given below without proof.

To calculate P_1 and K_1 , let $\ell = m-i+1$ and solve the following equations in sequence for $i = 1, 2 \dots m$.

$$\begin{bmatrix} p_{\ell\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_1 = \begin{bmatrix} k_{\ell\ell} & \dots & k_{\ell m} \\ \vdots & & \\ k_{m\ell} & \dots & k_{mm} \end{bmatrix}^{-1} \begin{bmatrix} r_{\ell\ell} \\ \vdots \\ r_{m\ell} \end{bmatrix} \quad (5.11)$$

$$\begin{bmatrix} k_{1\ell} \\ \vdots \\ k_{\ell-1\ell} \end{bmatrix}_1 = \frac{1}{(p_{\ell\ell})_1} \left\{ \begin{bmatrix} r_{1\ell} \\ \vdots \\ r_{\ell-1\ell} \end{bmatrix}_1 - \begin{bmatrix} k_{1\ell+1} & \dots & k_{1m} \\ \vdots & & \\ k_{\ell-1\ell+1} & \dots & k_{\ell-1m} \end{bmatrix} \begin{bmatrix} p_{\ell+1\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_1 \right\} \quad (5.12)$$

The use of Equations (5.11) and (5.12) in solving for P_1 and unknown part of k_1 is further demonstrated via example of Section 5.5.

To solve for the unknown parts of P_j and K_j , let $\ell = m-i+1$ and proceed in sequence for $i = 1, 2 \dots m$.

If at any point $[p_j^\ell]$ is completely specified, then determine $[k_j^\ell]$ from the following

$$[k_j^\ell] = \frac{1}{(p_{\ell\ell})_1} \left\{ \begin{bmatrix} k_1^\ell & \dots & k_1^m \\ \vdots & & \\ p_{m\ell} \end{bmatrix}_j - [q_r^\ell] - [k_j^{\ell+1} \dots k_j^m] \begin{bmatrix} p_{\ell+1\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_1 \right\} \quad (5.13)$$

Otherwise, calculate unspecified parts of p_j^ℓ for corresponding arbitrary elements in k_j^ℓ as follows:

$$\begin{bmatrix} p_{\ell+1,\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_j = \begin{bmatrix} k_{\ell+1\ell+1} & \dots & k_{\ell+1m} \\ \vdots & & \vdots \\ k_{m\ell+1} & \dots & k_{mm} \end{bmatrix}_1^{-1} \left\{ \begin{bmatrix} q_{\ell+1\ell} \\ \vdots \\ q_{m\ell} \end{bmatrix}_r + \begin{bmatrix} k_{\ell+1\ell} & \dots & k_{\ell+1m} \\ \vdots & & \vdots \\ k_{m\ell} & \dots & k_{mm} \end{bmatrix}_j \begin{bmatrix} p_{\ell\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_1 - \begin{bmatrix} k_{\ell+1\ell} \\ \vdots \\ k_{m\ell} \end{bmatrix}_1 (p_{\ell\ell})_j \right\} \quad (5.14)$$

and, then, finally find unspecifiable parts of k_j^ℓ using the following equation.

$$\begin{bmatrix} k_{1\ell} \\ \vdots \\ k_{\ell\ell} \end{bmatrix}_j = \frac{1}{(p_{\ell\ell})_1} \left\{ \begin{bmatrix} k_{1\ell} & \dots & k_{1m} \\ \vdots & & \vdots \\ k_{\ell\ell} & & k_{\ell m} \end{bmatrix}_1 \begin{bmatrix} p_{\ell\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_j - \begin{bmatrix} q_{1\ell} \\ \vdots \\ q_{\ell\ell} \end{bmatrix}_r - \begin{bmatrix} k_{1\ell+1} & \dots & k_{1m} \\ \vdots & & \vdots \\ k_{\ell\ell} & & k_{\ell m} \end{bmatrix}_j \begin{bmatrix} p_{\ell+1\ell} \\ \vdots \\ p_{m\ell} \end{bmatrix}_1 \right\} \quad (5.15)$$

In all of the above equations $\ell = m-i+1$, and one must proceed sequentially for $i = 1, 2 \dots m$ as shown in the example of Section 5.5.

The proof of the theorem is thus complete. In the next section a step-by-step design procedure is outlined which is used in a subsequent example.

5.4 Summary of the Design Procedure

The statement and proof of Theorem 5.1 has resulted in design constraints and design procedure. The constraints are summarized below and are followed by step-by-step design procedure for quick reference.

The main advantage of the design method described in this section is that all the n poles of overall system are specified (fixed) in advance by forcing $P(s)$ a triangular matrix and by specifying all diagonal terms to within a constant. In addition, if $R(s)$ meets certain prespecified conditions, then additional error coefficients can be specified for those terms in $T(s)$ for which corresponding terms in $P(s)$ are free. Thus, if $P(s)$ is forced to be a lower (upper) triangular matrix, then one has the following information:

1. The terms above (below) the diagonal terms in $P(s)$ are forced to be zero and, hence, for corresponding above (below) the diagonal terms in $T(s)$ no error coefficients can be specified.
2. The diagonal terms are fixed to within a constant, and thus these constants can be used to specify the first error coefficients (position error coefficients) of the diagonal terms in $T(s)$.
3. The terms below (above) the diagonal terms in $P(s)$ are free polynomials of degree $\leq \sigma_i$ for the i th column in $P(s)$, and hence can be used to specify σ_i more error coefficients for the terms in i th column of $T(s)$.

Thus, for each component of $P(s)$ except for P_1 only the terms below (above) the diagonal are unknown and all other terms are known. For

P_1 all terms below (above) and including the diagonal terms are unknown and the rest are identically zero. For K_1 all terms below (above) and including the diagonal terms can be specified as desired provided they are specified in such a way as to force K_1 lower (upper) definite. For K_j , $j \geq 2$ only terms corresponding to the free elements in P_j can be specified and the rest must be determined.

The step-by-step procedure below utilizes the above information and summarizes the steps involved in the design.

Steps 1-5: These steps are exactly the same as those given for the design method of the last chapter. These consist of deriving the feedback invariants of the system.

Step 6: Break $R(s)$ in its components and examine R_1 . If R_1 is lower (upper) definite, choose $P(s)$ lower (upper) triangular. If R_1 is definite nonsingular $P(s)$ could be chosen either lower or upper triangular, whichever form is more useful. Go to Step 8.

Step 7: If R_1 is neither upper triangular nor lower triangular, then a change in the names of the inputs or outputs might do the job in most cases. If so, repeat Steps 1 through 6.

Step 8: Determine P_1 and the unspecifiable portion of K_1 by using Equations (5.3) and (5.4) as described in the last section. [Use Equations (5.11) and (5.12) for lower triangular $P(s)$.]

Step 9: Determine the diagonal terms of $P(s)$ from the knowledge of the diagonal terms in P_1 . The diagonal terms in $P(s)$ are specified to within a constant and these constants can be determined by comparing them with the diagonal terms of P_1 .

Step 10: Find the complete P_j and K_j by using Equations (5.8), (5.9) and (5.10) and solving for $i = 1, 2 \dots m$ in sequence. [Use Equations (5.13), (5.14), and (5.15) for $P(s)$ lower triangular.] Solve for $j = 1, 2 \dots$ until all components of P_j are completely known.

Step 11: Find the closed loop transfer matrix $T(s) = R(s)P^{-1}(s)$ as both $R(s)$ and $P(s)$ are now completely known.

Step 12: Find the l.s.v.f. pair $\{F, G\}$ to realize this $T(s)$ using Eq. (2.36).

This completes the design. In the next section the example of the last chapter is reworked using the new design method of this chapter.

5.5 Example 5.1

Consider the same system as in the example of the last chapter. Since Steps 1 through 5 in the pole fixing method are exactly the same as in design method of the last chapter, the results of these steps are summarized below. For details, see Section 4.5.

Step 1: $\sigma_1 = 2, \sigma_2 = 1$

Step 2: The transformation matrix Q is

$$Q = \begin{bmatrix} -0.25 & 0.125 & 0 \\ 1.25 & -0.125 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$$

Step 3:

$$S^{\sigma} = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}, \quad S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

Step 4:

$$R(s) = \hat{C} S(s) = \begin{bmatrix} s+1 & 0 \\ 2(s+5) & 1 \end{bmatrix}$$

$$P(s) = \begin{bmatrix} d_{11}s^2 + h_{12}s + h_{11} & d_{12}s + h_{13} \\ d_{21}s^2 + h_{22}s + h_{21} & d_{22}s + h_{23} \end{bmatrix}$$

Step 5: Break the elements of the desired transfer matrix in their Maclaurin series. This is done in Section 4.5, and is not repeated here.

Step 6:

$$R_1 = R(s) \Big|_{s=0} = \begin{bmatrix} s+1 & 0 \\ 2(s+5) & 1 \end{bmatrix} \Big|_{s=0} = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix} \quad (5.16)$$

Similarly,

$$R_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad R_3 = R_4 = \dots = 0$$

By inspection, R is definite nonsingular matrix and, hence, is both lower and upper nonsingular (Def. 5.1). $P(s)$ could thus be forced either lower or upper definite. As mentioned earlier, this is an added advantage. Since $\sigma_1 = 2$ and $\sigma_2 = 1$, forcing $P(s)$ to be lower triangular results in greater design freedom because for $t_{21}(s)$, $\sigma_1 + 1 = 3$. Hence, three error coefficients can be chosen. By forcing $P(s)$ upper triangular, only $\sigma_2 + 1 = 2$ error coefficients can be freely specified for $t_{12}(s)$. So unless realization of $t_{12}(s)$ is more important than the realization of $t_{21}(s)$, $P(s)$ should be chosen lower triangular. For this example $P(s)$ is forced to be a lower triangular matrix.

As before, assume that the transfer matrix of Eq. (4.29) is still the desired transfer matrix but now the emphasis is on the poles of diagonal terms. It is desired that the overall system poles be the poles of $t_{11}(s)$ and $t_{22}(s)$ in Eq. (4.29). Since $P(s)$ is forced to be lower triangular, the above is realized by the following choice of $P(s)$

$$P(s) = \begin{bmatrix} d_{11}(s+2)(s+3) & 0 \\ d_{21}s^2 + h_{22}s + h_{21} & d_{22}(s+4) \end{bmatrix} \quad (5.17)$$

In the above d_{ij} and h_{ij} are as defined in Step 4.

The desired K_i 's are given by Eq. (4.30). But, as mentioned in Theorem 5.1, choice of the pole positions restricts the allowable freedom of the K_i 's as follows:

1. only the elements in K_1 for which $i \geq j$ can be chosen arbitrarily;
2. for (i,j) th element in the transfer matrix, only σ_j additional error coefficients can be chosen arbitrarily for $i > j$.

Thus, the specifiable part of the desired K_i 's are given below

$$k_1 = \begin{bmatrix} 1 & * \\ 1 & 1 \end{bmatrix}, \quad k_2 = \begin{bmatrix} * & * \\ -7 & * \end{bmatrix}, \quad k_3 = \begin{bmatrix} * & * \\ -.39 & * \end{bmatrix} \quad (5.18)$$

where * indicates unspecifiable element.

Step 7:

Since R_1 is nonsingular definite, this step is not necessary.

Step 8:

Since $P(s)$ is forced to be a lower triangular matrix, Eq. (5.11) and (5.12) are used to solve for P_1 and K_1 . Also, since $m=2$, these equations need be solved only for $i = 1, 2$ but in sequence.

Solving Eq. (5.11) for $i = 1$ gives $\ell = m-i+1 = 2$ and hence

$$\begin{aligned} (p_{22})_1 &= (k_{22})_1^{-1} (r_{22})_1 \\ &= (1.0)^{-1} (1.0) = 1.0 \end{aligned}$$

and solving Eq. (5.12) for $i = 1$ gives $\ell = m-i+1 = 2$. Hence

$$(k_{12})_1 = \frac{1}{(p_{22})_1} \{(r_{12})_1\} = \frac{1}{1} \{0\} = 0$$

For $i = 2$, $\ell = m-i+1 = 1$, and hence from Eq. (5.11)

$$\begin{aligned} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}_1 &= \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_{11} \\ r_{21} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \end{aligned}$$

Thus, P_1 and K_1 are completely known and are given as follows

$$P_1 = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad (5.19)$$

Step 9:

From Eq. (5.17),

$$P_1 = P(s) \Big|_{s=0} = \begin{bmatrix} 6d_{11} & 0 \\ h_{21} & 4d_{22} \end{bmatrix}$$

If this matrix is compared with Eq. (5.19), the following is readily obtained

$$d_{11} = 1/6, \quad d_{22} = 1/4, \quad h_{21} = 9$$

Substituting in Eq. (5.17) completely specifies the diagonal terms of $P(s)$ as follows

$$\begin{aligned}
 P(s) &= \begin{bmatrix} 1/6 (s+2)(s+3) & 0 \\ d_{21}s^2 + h_{22}s + 9 & 1/4(s+4) \end{bmatrix} \\
 &= \begin{bmatrix} 1 + 5/6 s + 1/6 s^2 & 0 \\ 9 + h_{22}s + d_{21}s^2 & 1 + 1/4 s \end{bmatrix}
 \end{aligned}$$

Hence

$$P_1 = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} .83 & 0 \\ h_{22} & .25 \end{bmatrix}, \quad P_3 = \begin{bmatrix} .17 & 0 \\ d_{21} & 0 \end{bmatrix} \quad (5.20)$$

Step 10:

Once again, since $P(s)$ is forced to be a lower triangular matrix, Eq. (5.13), (5.14) and (5.15) are used to determine the remaining unknown and unspecifiable components of $P(s)$ and $T(s)$.

To find P_2 and K_2 , solve those equations for $j=2$. For $i = 1$, $l = m-i+1 = 2$. But $[p_2^2]$ is completely known from Eq. (5.19) and, hence $[k_2^2]$ is found directly from Eq. (5.13).

$$[k_2^2] = \frac{1}{(p_{22})_1} \{ [k_1^2] (p_{22})_2 - [q_1^2] \}$$

Also, from Eq. (3.19) and (5.16)

$$Q_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

Hence,

$$[k_2^2] = \frac{1}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (.25) = \begin{bmatrix} 0 \\ .25 \end{bmatrix} \quad (5.21)$$

For $i = 2$, $\ell = m-i+1 = 1$ and hence from Eq. (5.14)

$$(p_{21})_2 = (k_{22})_1^{-1} \left\{ (q_{21})_1 + [k_{21} \quad k_{22}]_2 \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}_1 - (k_{21})_1 (p_{11})_2 \right\}$$

Notice that $(k_{22})_2$ has been obtained in Eq. (5.21), $(k_{21})_2$ is the arbitrary (desired) error coefficients specified in Eq. (5.18) and all other elements quantities on the right are known. Hence

$$\begin{aligned} (p_{21})_2 &= (1)^{-1} \left\{ 2 + [.7 \quad .25] \begin{bmatrix} 1 \\ 9 \end{bmatrix} - (1)(.83) \right\} \\ &= 4.12 \end{aligned}$$

Finally, the unknown part of K_2 is determined by solving Eq. (5.15)

for $i = 2$.

For $i = 2$, $\ell = m-i+1 = 1$ and hence

$$\begin{aligned}
 (k_{11})_2 &= \frac{1}{(p_{11})_1} \left\{ [k_{11} \quad k_{12}] \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} - (q_{11})_1 - (k_{12})_2 (p_{21})_1 \right\} \\
 &= \frac{1}{1} \left\{ [1 \quad 0] \begin{bmatrix} .83 \\ 4.12 \end{bmatrix} - (1) - (0)(9) \right\} \\
 &= .83 - 1 = - .17
 \end{aligned}$$

The P_2 and K_2 matrices are known completely as given below

$$P_2 = \begin{bmatrix} .83 & 0 \\ 4.12 & .25 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -.17 & 0 \\ 0.7 & .25 \end{bmatrix} \quad (5.22)$$

At last, proceed to find elements in P_3 and K_3 . To do that, first calculate Q_2 as given by Eq. (3.19)

$$\begin{aligned}
 Q_2 &= R_3 + K_2 P_2 \\
 &= 0 + \begin{bmatrix} -.17 & 0 \\ 0.7 & .25 \end{bmatrix} \begin{bmatrix} .83 & 0 \\ 4.12 & .25 \end{bmatrix} = \begin{bmatrix} -.139 & 0 \\ 1.613 & .062 \end{bmatrix} \quad (5.23)
 \end{aligned}$$

For $i = 1$, $\ell = m-i+1 = 2$ and since $[p_3^2]$ is known, $[k_3^2]$ is directly calculated by solving Eq. (5.13) for $j = 3$

$$\begin{aligned}
[k_3^2] &= \frac{1}{(p_{22})_1} \{ [k_1^2] (p_{22})_3 - [q_2^2] \} \\
&= \frac{1}{1} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) - \begin{bmatrix} 0 \\ .062 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ -.062 \end{bmatrix}
\end{aligned}$$

For $i = 2$, $\ell = m-i+1 = 1$ and the unspecified part of P_3 may be calculated by solving Eq. (5.14) for $j = 3$

$$\begin{aligned}
(p_{21})_3 &= (k_{22})_1^{-1} \left\{ (q_{21})_2 + [k_{21} \quad k_{22}]_3 \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}_1 - (k_{21})_1 (p_{11})_3 \right\} \\
&= (1)^{-1} \left\{ 1.613 + [-.39 \quad -.062] \begin{bmatrix} 1 \\ 9 \end{bmatrix} - (1)(.17) \right\} \\
&= 1.613 - .39 - .5625 - .17 = .491
\end{aligned}$$

Notice that the solution of the above equation was made possible by ensuring the inverse of $(k_{22})_1$. Elements in K_1 were so chosen as to force K_1 as obtained in Eq. (5.19) to be a lower definite matrix.

Finally, after substitution of the results obtained so far, the only unknown part of K_3 is obtained by solving Eq. (5.15) for $j = 3$ and $i = 2$. For $i = 2$, $\ell = m-i+1 = 1$ and hence

$$\begin{aligned}
(k_{11})_3 &= \frac{1}{(p_{11})_1} \left\{ [k_{11} \quad k_{12}]_1 \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}_3 - (q_{11})_2 - (k_{12})_3 (p_{21})_1 \right\} \\
&= \frac{1}{1} \left\{ [1 \quad 0] \begin{bmatrix} .17 \\ .491 \end{bmatrix} - (-.139) - (0)(9) \right\} \\
&= .17 + .139 = .31
\end{aligned}$$

Thus, P_3 and K_3 are completely known as

$$P_3 = \begin{bmatrix} .17 & 0 \\ .491 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} .31 & 0 \\ -.39 & -.062 \end{bmatrix} \quad (5.24)$$

The design is almost complete, except that one has to find the pair $\{F, G\}$ which corresponds to the components of $P(s)$ obtained in Equations (5.19), (5.22), and (5.24). This can be trivially done as was done for the example (4.1). However, the first order of business is to check whether the transfer matrix so obtained is satisfactorily close to the one wanted. This is done as follows:

$$\begin{aligned}
T(s) &= R(s)P(s)^{-1} \\
&= \begin{bmatrix} s+1 & 0 \\ 2(s+5) & 1 \end{bmatrix} \begin{bmatrix} .17(s^2 + 5s + 6) & 0 \\ .491s^2 + 4.12s + 9 & .25(s+4) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} s+1 & 0 \\ 2(s+5) & 1 \end{bmatrix} \begin{bmatrix} .25(s+4) & 0 \\ -(.491s^2 + 4.12s + 9) & .17(s^2 + 5s + 6) \end{bmatrix} / \Delta
\end{aligned}$$

where $\Delta = 1/24 (s+2)(s+3)(s+4)$.

Hence

$$\begin{aligned}
 T(s) &= \begin{bmatrix} \frac{6(s+1)}{(s+2)(s+3)} & 0 \\ \frac{24(.009s^2 + .38s + 1)}{(s+2)(s+3)(s+4)} & \frac{4}{s+4} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{6(s+1)}{(s+2)(s+3)} & 0 \\ \frac{.216s^2 + 9.02s + 24}{(s+2)(s+3)(s+4)} & \frac{4}{s+4} \end{bmatrix} \quad (5.25)
 \end{aligned}$$

Once again the transfer matrix obtained above could be compared with the one desired as given in Eq. (4.29), and repeated here for convenience.

$$T(s) = \begin{bmatrix} \frac{6(s+1)}{(s+2)(s+3)} & \frac{0.6(s+1)}{(s+2)(s+3)} \\ \frac{10}{(s+2)(s+5)} & \frac{4}{s+4} \end{bmatrix} \quad (5.26)$$

Since the emphasis is on the poles of the diagonal terms, one can see they are exactly at the desired place. However, this is possible only by the sacrifice of the term $t_{12}(s)$ in Eq. (5.26). The situations in which the off diagonal terms come out to be zero is a special case, as is discussed in the next section. The term $t_{21}(s)$ in the transfer matrix of Eq. (5.25) has the same generalized error coefficients as specified by the corresponding term in desired transfer matrix in Eq. (5.26).

However, the pole positions of $t_{21}(s)$ are not the same as desired. It is up to the designer to test this transfer function to see if its transient response is satisfactory and within his tolerance limits. The step responses of the two are compared in Fig. 5.1.

If the transfer matrix obtained above is satisfactory, the pair $\{F, G\}$ necessary to realize it are easily found by proceeding on exactly the same lines as was done for example 4.1. Here, only the results are given for the sake of completeness.

$$F = \begin{bmatrix} 1.5 & -.25 & 0 \\ -2.21 & -2.19 & -2 \end{bmatrix}$$

$$G = \begin{bmatrix} 6 & 0 \\ -11.8 & 4 \end{bmatrix} \quad (5.27)$$

In the next section some special case are discussed.

5.6 Special Cases

5.6.1 The Case of Triangular Decoupling

It is seen in Example 5.1 that by fixing the poles in advance, the resulting transfer matrix came out to be lower triangular, even though it was not intended so. This may have certain advantages if it could be predicted in advance. The following lemma gives the sufficiency condition for triangular decoupling.

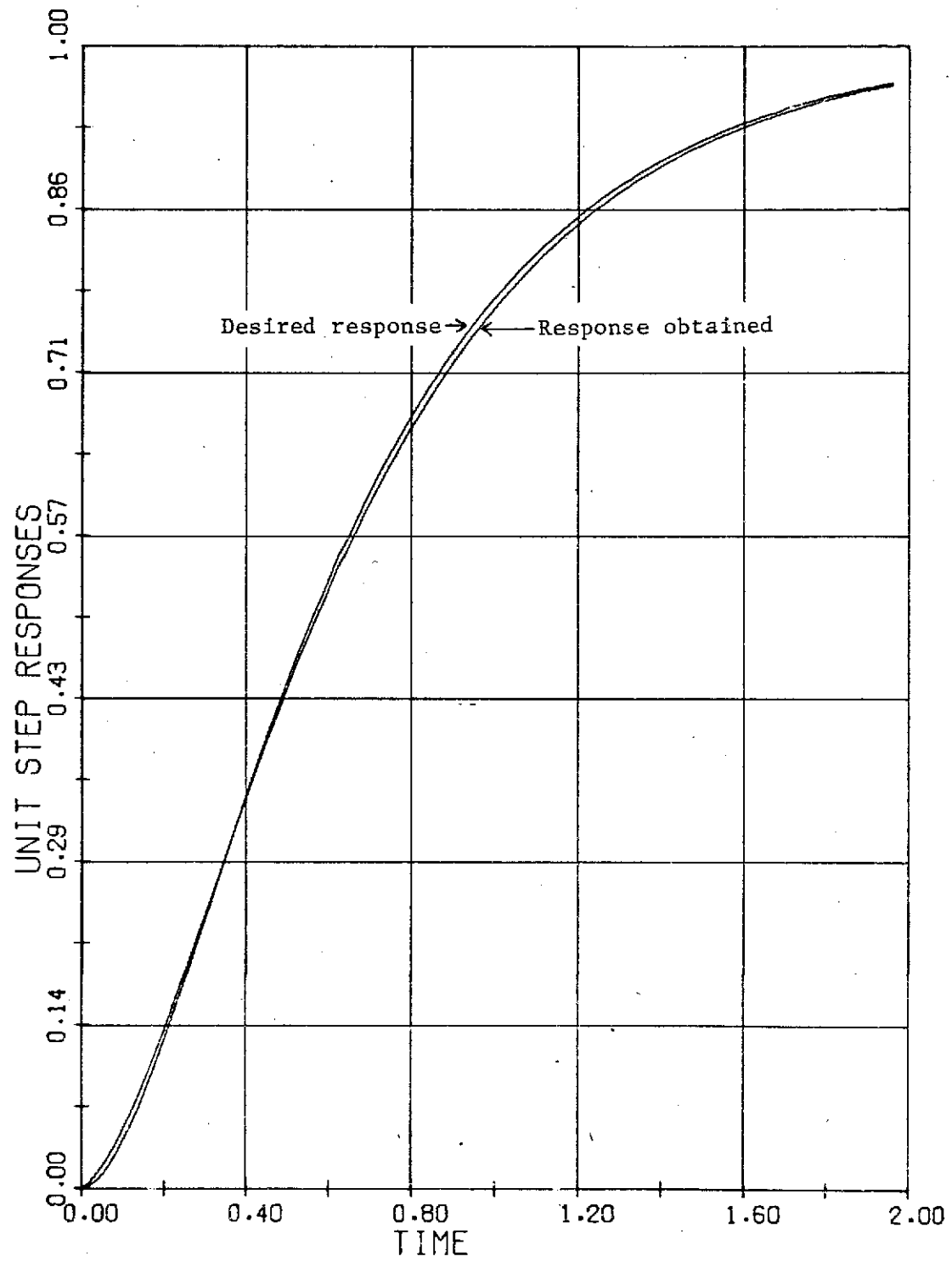


Fig. 5.1 Comparison of Time Responses in Example 5.1

Lemma 5.1 If $R(s)$ is upper (lower) triangular, then $T(s)$ is upper (lower) triangular decoupled by choosing $P(s)$ upper (lower) triangular, provided $R_1 = R(s)|_{s=0}$ is nonsingular.

Proof: If $P(s)$ is upper (lower) triangular, then $P^{-1}(s)$ is also upper (lower) triangular, and hence $T(s) = R(s)P^{-1}(s)$ is also upper (lower) triangular. This together with Theorem 5.1 proves the lemma because $R(s)$ triangular and R_1 nonsingular implies R_1 is absolutely nonsingular [Definition 5.1 in Section 5.2].

The main advantage gained by above is that instead of having no control over the one side of the diagonal transfer functions (Theorem 5.2), they can be forced to be zero.

Since $R(s)$ is completely known in advance before any actual design is attempted, it could be easily found whether $R(s)$ is a triangular matrix or can be made triangular by simple changes in the names of inputs and outputs. Particularly if the open loop system is coupled in one direction only, then a change in the names of inputs and outputs would make $R(s)$ a triangular matrix. Of course, a change in the output matrix can cause a significant change in $R(s)$. If such a change is not intolerable, then it might be worth forcing $R(s)$ triangular.

Thus, if $R(s)$ is triangular, then the designer knows in advance that by using the pole fixing method, a triangular transfer matrix design can be obtained. This has distinct advantage over conventional decoupled

design where only diagonal transfer functions are nonzero and all other transfer functions are forced to zero.

5.6.2 The Minimum Constraint Case

The case in which there is a minimal constraint on the generalized error coefficient is stated through the following corollary.

Corollary 5.1: If all σ 's are equal, i.e., $\sigma_1 = \sigma_2 = \dots = \sigma_m = \sigma$, then for each transfer function in the transfer matrix $T(s)$, $\sigma + 1$ arbitrary generalized error coefficients can be realized if:

1. K_1 is chosen nonsingular. (This is a necessary condition for the system to be stable and to have nonsingular transfer matrix as stated in Theorem 3.1.)
2. $K_{\sigma+1}$ is chosen so that $P_{\sigma+1}$ is nonsingular.

Proof: The proof follows from Theorem 4.1. In addition, note that for any choice of $P_{\sigma+1}$, $K_{\sigma+1}$ is uniquely determined via Eq. (3.19), since K_1 and, hence, P_1 are nonsingular. This guarantees that there are many sets of $K_{\sigma+1}$ which result in nonsingular $P_{\sigma+1}$.

This case of equal σ 's is mentioned here for its academic value only. In practice, the σ 's, even though non-unique are feedback invariants. Thus, if n is not an integral multiple of m , it is impossible to make them equal without adding dynamics to the system. The case in which dynamics are added to the system is not considered

in this study and in fact, has been considered by others, the most recent of which is the work of Wolowich [1973].

5.7 Summary

In this chapter a new design method called the pole fixing method is used for the synthesis of multivariable systems. In many cases the pole fixing method can be successfully used for the design of multivariable systems where the conventional design methods or the error coefficient method of the last chapter fails. In this method all the poles of the multivariable system are pre-assigned and the zeros are adjusted to produce satisfactory steady state behavior. The only disadvantage of the pole fixing method is that it puts new constraints on the feedback invariant part of the system, $R(s)$. But it is seen that in many cases, simple changes in the output matrix satisfy the artificial constraints introduced by the pole fixing design method.

CHAPTER 6

SUMMARY AND CONCLUSION

This chapter summarizes the results of this study and pertinent results concerning linear state variable feedback (l.s.v.f.) invariants. Some areas of further research in connection with this study are also indicated.

6.1 Summary

Linear state variable feedback has been used in the design of multivariable systems. Multivariable systems differ from the scalar case in that l.s.v.f. affects not only the pole positions, but also directly affects the zeros because of the coupling between the subsystems.

The l.s.v.f. invariants of multivariable systems were not known completely until recently. Wolowich [1972] introduced a complete set of l.s.v.f. invariants for multivariable systems for the first time. He utilized the structure of multivariable systems to derive l.s.v.f. invariants and, in turn, used l.s.v.f. invariants to derive the necessary and sufficient condition for exact model matching. If l.s.v.f. can be found, such that the closed loop response matches a pre-specified (model) response, then the plant (open loop system) is said to match with the model.

The necessary and sufficient conditions for exact model matching did not come as a relief to the designer. The shortcoming is that if the necessary and sufficient conditions are not met, there is no way to complete the design. One does not know what changes need to be made in the design specification to satisfy the necessary and sufficient conditions. Thus, the problem of designing a multivariable system using l.s.v.f. where noninteraction is not a design criteria remained largely unanswered. An exception is a very special case where the dynamics of each element in any given row of the transfer matrix are the same except for the gain [Ferg, 1971].

In this study two complete design procedures are developed for the first time for the design of multivariable systems incorporating cross coupling using l.s.v.f. alone. Noninteraction is treated as a special case of interaction. The main design equation is developed in Chapter 3. The first design procedure using error coefficients alone as the design criteria is worked out in Chapter 4. This is followed by a simple example to illustrate the method.

The generalized error coefficients represent the system input-output relationship only after the system transients have decayed to insignificant amplitudes. Here lies the shortcoming of the error coefficient design method, since it does not take into account the transient behavior of the system. To alleviate this problem the pole fixing method is introduced. As the name suggests, all the poles of the system are fixed in advance ensuring system stability and fast

decaying transients. If the system satisfies certain sufficiency conditions, then in addition to fixing all the poles at desired places, some indirect control can be exercised over zeros. In addition, those systems which cannot be decoupled can be designed using the pole fixing method. Obviously, this method of design does not produce any uncontrollable poles as does the noninteracting design.

The sufficiency conditions are variant under the change in system output matrix. Thus, in many cases a simple operation like changing the name of inputs or outputs could be enough to satisfy the sufficiency conditions. The application of the pole fixing method is illustrated by working a simple example.

6.2 Further Research

Although this study provides two design methods for the design of multivariable systems. The following related topics merit further research:

1. Establish additional definite relationships between the specifiable error coefficients, the poles of the overall system, and the component matrices of the matrix $P(s)$. The first of the two design methods for the design of multivariable systems is based upon the assumption that the system input-output behavior can be satisfactorily described in terms of finite error coefficients. In many cases this may not be true. In particular, when a transfer function is expanded in terms of error coefficients, much of the information about the transient response is

lost because only a relatively small number of error coefficients are considered. The transfer function cannot always be uniquely reconstructed only from the specification of these error coefficients.

2. Investigate the relation between the components of a polynomial matrix and its determinant. This is useful because poles of the overall system are given by the zeros of determinant $P(s)$. The components of $P(s)$ are determined one by one. The relation between the components of $P(s)$ and its determinant can then be used to determine whether the particular component of $P(s)$ is suitable enough for the design to proceed any further. If not, the error coefficients could be suitably changed before the design is complete. For example, the constant term in the determinant of $P(s)$ is given by the determinant of P_1 , the first component of $P(s)$.
3. Partially or completely fix the poles of the systems without forcing $P(s)$ a triangular matrix. By forcing $P(s)$ a triangular matrix the designer could of course fix all the poles of the system in advance. But this has the disadvantage that control is lost over the zeros of one side of diagonal terms in $T(s)$. Also, fixing all poles adds additional constraints which must be satisfied. Thus, forcing $P(s)$ to be triangular might be over restrictive. Two obvious ways to proceed are to either partially fix the poles or equivalently to find a way in which $P(s)$ is not forced to be a triangular matrix.

4. Find the necessary condition for the design method to be applicable. Theorem 4.1 and 5.1 provide only sufficient conditions to complete the design. Upon investigation it might turn out that some of the sufficiency conditions are also necessary conditions. The advantage of knowing the necessary conditions is that the designer knows that he is not being too conservative in designing the system.

It is seen that even though linear state variable feedback (l.s.v.f.) can be successfully used for the design of multivariable systems, all of its implications in the design of multivariable feedback have not been resolved. It is expected that efforts of more than just a few will be required before those working in the area feel that the problem is solved.

SELECTED BIBLIOGRAPHY

- Agrawal, B. L. "Design of Noninteracting Multivariable Discrete Time Systems," Master's Thesis, Univ. of Ariz., 1972.
- Anderson, B. D. O., and D. G. Luenberger. "Design of Multivariable Feedback Systems," IEE Proceedings, Vol. 114, March, 1967.
- Boksenbom, A. S., and R. Hood. "General Algebraic Method Applied to Control Analysis of Complex Engine Types," NASA Technical Report, TR 980, April, 1949.
- Brockett, R. W. "Poles, Zeros, and Feedback: State Space Interpretation," IEEE Trans. Automatic Control, Vol. AC-10, April, 1965.
- Chen, C. T., and C. H. Hsu. "Design of Dynamic Compensators for Multivariable Systems," Joint Automatic Control Conference, St. Louis, Paper No. 8E-1, 1971.
- Davison, E. J. "On Pole Assignment in Multivariable Linear Systems," IEEE Trans. Automatic Control, Vol. AC-13, December, 1968.
- Davison, E. J., and S. G. Chow. "An Algorithm for the Assignment of Closed Loop Poles Using Output Feedback in Large Linear Multivariable Systems," IEEE Trans. Automatic Control, Vol. AC-18, February, 1973.
- Falb, P. L., and W. A. Wolowich. "Decoupling in the Design and Synthesis of Multivariable Control Systems," IEEE Trans. Automatic Control, Vol. AC-12, December, 1967.
- Ferg, D. A. "Multiple Input-Output Feedback Synthesis Incorporating Cross Coupling," Ph.D. Dissertation, University of Arizona, Tucson, Arizona, 1971.
- Gantmacher, F. R. The Theory of Matrices, Vol. I, Chelsea Publishing Company, New York, 1959.
- Gilbert, E. G. "The Decoupling of Multivariable Systems by State Feedback," SIAM Journal Control, Vol. 2, No. 1, 1969.
- Heinz, E. "Analysis and Design of Model Following Control Systems by State Space Techniques," Joint Automatic Control Conference, Ann Arbor, Michigan, 1968.

- Horowitz, I. M. "Synthesis of Linear, Multivariable Feedback Control Systems," IRE Trans. Automatic Control, Vol. AC-5, June 1960.
- Kavanagh, R. J. "The Application of Matrix Methods to Multivariable Control Systems," Journal Franklin Institute, Vol. 262, November, 1956.
- Kavanagh, R. J. "Noninteracting Controls in Linear Multivariable Systems," AIEE Trans., Pt. II, Vol. 76, May, 1957.
- Kavanagh, R. J. "Multivariable Control System Synthesis," AIEE Trans., Pt. 11, Vol. 77, November, 1958.
- Luenberger, D. G. "Canonical Forms for Linear Multivariable Systems," IEEE Trans. Automatic Control, Vol. AC-12, June, 1967.
- MacDuffee, C. C. The Theory of Matrices, Chelsa Publishing Company, New York, 1956.
- Moore, B. C. and L. M. Silverman. "Exact Model Matching by State Feedback and Dynamic Compensation," Decision and Control Conference, Miami Beach, 1971.
- Morgan, B. S. "The Synthesis of Linear Multivariable Systems by State Variable Feedback," Preprints of Papers, Joint Automatic Control Conference, Stanford University, 1964.
- Morse, A. S., and W. M. Wonham. "Decoupling and Pole Assignment by Dynamic Compensation," SIAM Journal Control, Vol. 8, August, 1970.
- Pearson, J. B., and C. Y. Ding. "Compensator Design for Multivariable Linear Systems," IEEE Trans. Automatic Control, Vol. AC-14, April, 1969.
- Rekasius, Z. V. "Decoupling of Multivariable Systems Using State Variable Feedback," Third Allerton Conference, 1965.
- Rosenbrock, H. H. State Space and Multivariable Theory, Nelson and Sons, London, 1970.
- Savo, T. A. "Significance of the Generalized Error Coefficients in Servomechanism Synthesis," Master's Thesis, Purdue Univ., 1953.
- Schultz, D. G., and J. L. Melsa. State Functions and Linear Control Systems, McGraw-Hill Book Company, New York, 1967.

- Silverman, L. M., and H. J. Payne. "Input Output Structure of Linear Systems with Application to the Decoupling Problem," SIAM Journal Control, Vol. 9, May, 1971.
- Silvinsky, C. R. "The Design of Linear Multivariable Control Systems Using Modern Control Theory," Ph.D. Dissertation, University of Arizona, Tucson, Arizona, 1969.
- Simon, J. D., and S. K. Miller. "Synthesis of Transfer Function Matrices with Invariant Zeros," IEEE Trans. Automatic Control, Vol. AC-14, August, 1969.
- Sridhar, B., and D. P. Lindorff. "A Note on Pole Assignment," IEEE Trans. Automatic Control, Vol. AC-17, December, 1972.
- Truxal, J. G. Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, New York, 1955.
- Wang, S. H., and C. A. Desoer. "The Exact Model Matching of Linear Multivariable Systems," IEEE Trans. Automatic Control, Vol. AC-17, June, 1972.
- Wolowich, W. A. "A Direct Frequency Domain Approach to State Feedback and Estimation," IEEE Decision and Control Conference, Miami Beach, 1971.
- _____. "The Use of State Feedback for Exact Model Matching," SIAM Journal Control, Vol. 10, August 1972.
- _____. "On the Synthesis of Multivariable Systems," IEEE Trans. Automatic Control, Vol. AC-18, February, 1973.
- Wolowich, W. A., and P. L. Falb. "On the Structure of Multivariable Systems," SIAM Journal Control, Vol. 7, August, 1969.
- Wonham, W. M. "On Pole Assignment in Multivariable Controllable Linear Systems," IEEE Trans. Automatic Control, Vol. AC-12, December, 1967.